Test problems for Vlasov-Poisson

Andrew J. Christlieb [1,2] Yaman Güçlü [3] William N.G. Hitchon [4]
David Seal [1] John Verboncoeur[2]

[1]Dept. of Mathematics, Michigan State University, East Lansing (MI)

[2]Dept. of Electrical and Computer Engineering, Michigan State University, East Lansing (MI)

[3]Institute for Plasma Physics - Max-Planck-Institut, Garching, Germany

[4]Dept. of Electrical and Computer Engineering, University of Wisconsin, Madison (WI)

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Outline

1. Introduction
   - Vlasov-Poisson system for multi-species collisionless plasma
   - Numerical scheme employed
   - Bounded domains and sheaths
2. The collisionless sheath
   - Electrons only: ‘matrix sheath’ problem (Lieberman)
   - Maxwellian electrons and cold ions: Bohm criterion
   - General distribution functions: generalized Bohm criterion
3. Numerical experiments
   - Sheath at a floating wall (Kolobov et. al.)
   - ‘Source-collector’ system (Rizopoulou et. al.)
   - Improved boundary conditions
4. Conclusions and future work

Model equations (non-dimensional)

Poisson's equation:

$$\frac{\partial^2}{\partial x^2}\phi(t,x) = -\rho(t,x), \qquad E(t,x) = -\frac{\partial \phi}{\partial x}.$$

Charge density, multiple ion species α:

$$\rho(t,x) = \sum_{\alpha} Z_{i,\alpha} n_{i,\alpha}(t,x) - n_e(t,x)$$

Number density of each species:

$$n_e(t,x) = \int_{-\infty}^{+\infty} f_e(t,x,v)dv, \qquad n_{i,\alpha}(t,x) = \int_{-\infty}^{+\infty} f_{i,\alpha}(t,x,v)dv.$$

Vlasov's equation for each species (**collisionless plasma**):

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - E(t,x)\frac{\partial}{\partial v}\right) f_e(t,x,v) = 0,$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{Z_{i,\alpha} m_e}{m_{i,\alpha}} E(t,x)\frac{\partial}{\partial v}\right) f_{i,\alpha}(t,x,v) = 0, \qquad \forall \alpha$$

Numerical scheme

- Fully Lagrangian Methods
  - Contour Dynamics (Water Bag Method - D.C. DePackh 1962)
  - Particle Methods (PIC and P3M: Books- Birdsall and Langdon, Hockney and Eastwood)
  - Implicit Particle (Semi - A. Friedman, A. Langdon and B. Cohen 1981: Fully - G.Chen, L.Chacon, D.Barnes 2011)
- Semi Lagrangian Methods
  - Forward Lagrangian (W.N.G Hitchon 1987, E. Sonnendrucker 2012)
  - Backward Lagrangian (C.Z. Cheng, G. Knorr 1976)
- Eulerian Methods
  - Explicit (J. Hittinger, J. Banks 2010)
  - Implicit (Y. Cheng, A. J. Christlieb, X. Zhong, 2014)

Question: Which Method is Better?

Problem Dependent:

Quasi Neutral - Mesh Based Code?

Collisions and Rare Events - Mesh Based Code?

Beam problems - particle based code?

Zero Entry Pool - Periodic Test Problems

- Two Stream Instability
- Landau Damping
- MHD Pinch

Numerical scheme - An Example

- Eulerian representation of the distribution functions $f(t, x, v)$;
- $x \in [a, b]$, v-axis extent different for each species;
- Strang splitting [a]: Vlasov equation decomposed into 1D advection operators

$$\text{A:} \quad \frac{\partial f_e}{\partial t} = -v\frac{\partial f_e}{\partial x}, \qquad \text{B:} \quad \frac{\partial f_e}{\partial t} = E(t,x)\frac{\partial f_e}{\partial v},$$

$$f_e(t+\Delta t, x, v) = e^{\frac{\Delta t}{2}A} e^{\Delta t\, B} e^{\frac{\Delta t}{2}A} \cdot f_e(t,x,v) + O\left(\Delta t^3\right);$$

- Similarly for ions, no sub-cycling;
- Poisson's equation solved before operator B, using high order;
- Constant advection equations solved by high order backward characteristic [c] or mixed high order Explicit RK/backward characteristic [b];
- Overall algorithm is mass conservative [b,c], positivity preserving [b,c], energy-stable [c].

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[a] C. Cheng, and G. Knorr. *Journal of Computational Physics*, (1976).

[b] J. Rossmanith and D. Seal. *Journal of Computational Physics*, (2011). Discontinuous Galerkin - Seal 2D-2V PP

[c] Y. Güçlü, A.J. Christlieb, and W.N.G. Hitchon. *Journal of Computational Physics*, (2014). Arbitrarily Order FD PP

Bounded domain (1)

- In **periodic domains**, we study basic kinetic phenomena:
  - Collisionless damping, instabilities
  - Linear and non-linear regimes
- At **physical boundaries**, new phenomena take place:
  - Boundary layers (plasma sheath)
  - Absorbing walls (wall recombination of ions and electrons)
  - Surface charge buildup (dielectric wall, 'floating' electrode)
  - Net electric current (grounded, 'driven', and emissive electrodes)
- Also, **numerical boundaries** are useful:
  - Symmetry axes and planes
  - Truncated domain
- What do we need to simulate boundaries?
  - **Outflow** and **inflow** boundary conditions for constant advection
  - **Dirichlet** and **Neumann** boundary conditions for Poisson solver
  - Accumulate **surface charge** on wall (if any)

Bounded domain (2)

OUTFLOW:

- Mass must flow out of the system without spurious oscillations
- Because of dimensional splitting, small error near boundary may propagate back into system $\rightarrow$ **high order**

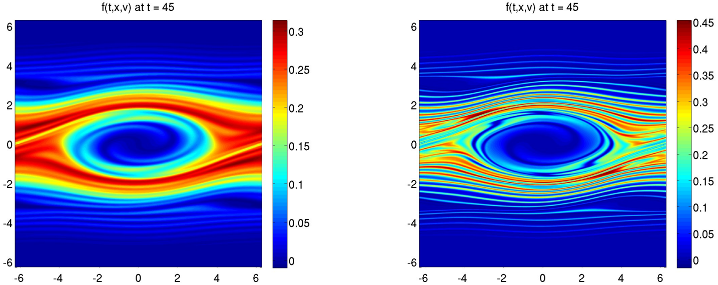
INFLOW - incoming distribution function may:

- Be zero (perfectly absorbing boundary)
- Be imposed externally (e.g. emissive electrode)
- Depend on outgoing distribution function (e.g. secondary electrons)

Zero Enerty Pool: Two-stream instability

2^{nd}-order Spatial Discretization

4^{th}-order Spatial Discretization



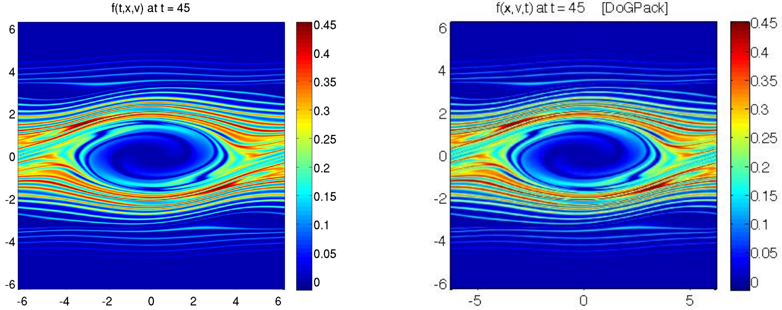
Exact Same Number of Unknows

- Uniform ion background $n_0 = 1$, charge neutrality at $t = 0$
- Periodic boundary conditions, $x \in \{-2\pi, 2\pi\}$
- $\alpha = 1$ for these runs. (big perturbation)

$$f_e(0, x, v) = \frac{v^2}{\sqrt{8\pi}} \left(2 - \alpha \cos\left(\frac{x}{2}\right)\right) e^{-\frac{v^2}{x}}$$

Fact: All High Order Methods Do About The Same!

4^{th} order Semi-Lagrangian | 4^{th} order Runge-Kutta



- SLWENO5 (Operator Split Approach): 13min on 2.5 GHz PC Qiu and Christlieb, JCP 2010
- SLDG4 (Operator Split Approach): time $\sim$ same J. Rossmanith, D. Seal, JCP 2011
- SLDG4 - PP (Operator Split Approach): time $\sim$ same Qiu and Shu, JCP 2011
- Arbitrary Order - PP (Operator Split Approach): time $\sim$ same Y. Guclu, A.J. Christlieb, W.N.G. Hitchon, JCP 2014
- SLWENO5 PP (Operator Split Approach): time $\sim$ same T. Xiong, J.-M. Qiu, Z. Xu, A.J. Christlieb, JCP accepted
- RKDG4 (Classical Approach): 14654m26.351s $\approx$ 72 hours

Test Problems?

- Algorithm and Model Verification and Validation (AMVV 2012)
  Test Problems With Boundaries
  http://www.egr.msu.edu/amvv2012/home
- Numerical Methods for Kinetic Equations of Plasmas Physics (NumKin 2013)
- Algorithm and Model Verification and Validation (AMVV 2014 April)
  http://www.ipp.mpg.de/amvv2014
- Algorithm and Model Verification and Validation (AMVV 2014 Oct)
- Gaseous Electronics Conference - Verification and Validation Workshop (GEC 2014 - Nov)

Laundry List

- Electrostatic Quasi Neutral
  Periodic, Bounded and Mixed
- Electrostatic Single Species
  Bounded and Mixed
- Electromagnetic Quasi Neutral
  Periodic, Bounded and Mixed, Relativistic or Non-Relativistic
- Electromagnetic Single Species
  Bounded and Mixed, Relativistic or Non-Relativistic

At What Density?, Gyro-averaged? . . .

Test Problem AMVV 2012 - Electron Beam

2D-2V, but has a 1D analog.

Goal:Building form the Zero Entry Pool

Initial conditions

$$f_e(0,x,y,v_x,v_y) = \frac{n_0}{(2\pi v_{th}^2)(\pi a^2)} \exp\left(-\frac{v_x^2+v_y^2}{2v_{th}^2}\right)\exp\left(-\frac{x^2+y^2}{2R^2}\right),$$

Modified equation: applied (confining) electric field [[a], [b], [c]]

$$f_{,t} + \mathbf{v}\cdot f_{,\mathbf{v}} - (\mathbf{E}+\mathbf{E}_a)\cdot f_{,\mathbf{v}} = 0, \qquad \mathbf{E}_a := -\omega_0^2\mathbf{x}$$

Self-consistent electric field

$$-\nabla^2\phi = -\rho(t,\mathbf{x}), \qquad -\nabla\phi = \mathbf{E}, \qquad \rho := \int_{\mathbf{v}} f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}.$$

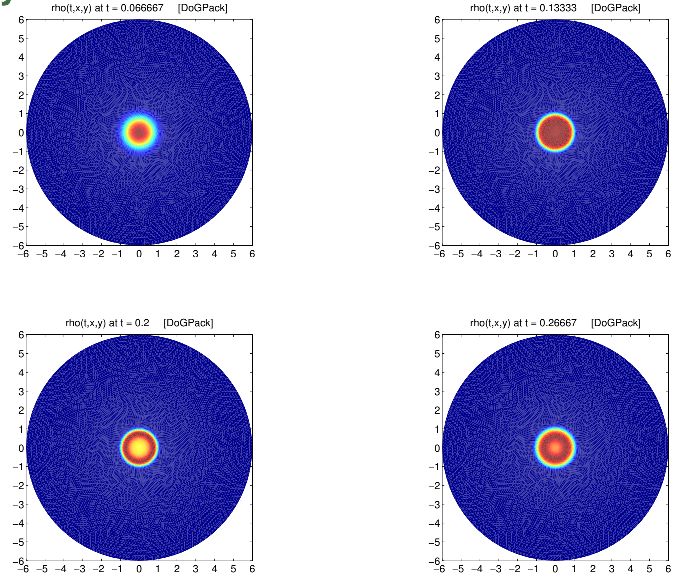
$\omega_0 = 8$, ω_0 frequency of external filed. $v_{thermal} = \frac{a\eta\omega_0}{2} = 1$, $R = \frac{1}{2}$ standard deviation of beam width and $a = 1$ length scale do to non-dimensionalzation. $R|_{\partial\Omega} = 6$

[a] N. BESSE, E. SONNENDRÜCKER *Journal of Computaional Physics*, (2003).

[b] N. BESSE, J. SEGRÉ, E. SONNENDRÜCKER *Journal of Computaional Physics*, (2005).

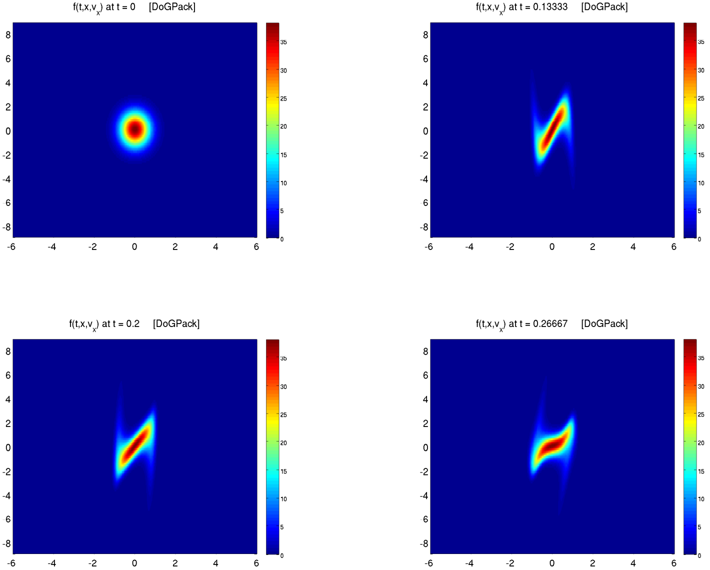
[c] R.C. DAVIDSON, H. QIN *Physics of Intense Charged Particle Beams in High Energy Accelerators.*, Imperial College Press, World Scientific (2001).

Density - Breathing Mode



10,000$\times$35$\times$35$\times$15=183,750,000 unknowns

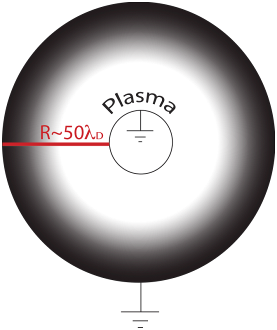
Phase space


f(t,x,v_x) at t = 0 [DoGPack]
f(t,x,v_x) at t = 0.13333 [DoGPack]
f(t,x,v_x) at t = 0.2 [DoGPack]
f(t,x,v_x) at t = 0.26667 [DoGPack]

Test Problem AMVV 2012 - Sheath

2D-2V: Building further form the Zero Entry Pool

- Quasi Neutral
- Sheath formation that can test a 2D-2V code, but has a 1D analog
- Investigate effect of stair steeping mesh in particle codes
- Detect asymmetries due to a mesh for Cut Cell/Finite Element methods
- Charge that hits the wall is absorbed



Electrons only: the matrix sheath problem [a]

- Uniform ion background $n_0 = 1$, charge neutrality at $t = 0$
- Symmetrical system, homogeneous Dirichlet BCs for potential
- Perfectly absorbing walls, no emission
- Electrons flow out of system and leave positive charge behind $\rightarrow$ sheath formation, **electron plasma oscillations**

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla_x + \nabla\phi \cdot \nabla_v\right) f_e(t,x,v) = 0$$

$$\nabla^2\phi = \int_{-\infty}^{+\infty} f_e(t,x,v)\,dv - 1$$

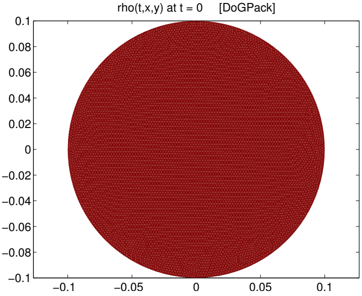
$$\phi(\partial\Omega) = 0$$

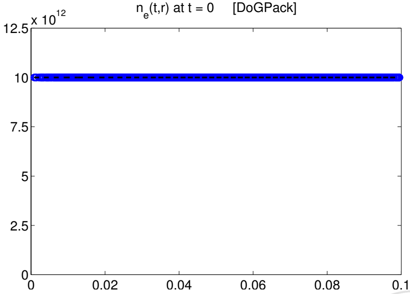
- Domain size $L = 50\lambda_D$
- Initial conditions $f_e(0,x,v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right)$
- Simulation length $50T_p = 50(2\pi/\omega_p)$

[a] M.A. Lieberman, A.J. Lichtenberg. **Principles of Plasma Discharges and Materials Processing**, 2nd Ed., *Wiley 2005*.

Plasma Sheath (2D-2V) with High Order SLDG

- Initial conditions: constant density, $n_e(0, \mathbf{x}) = n_i(t, \mathbf{x}) = 9.10938188 \times 10^{-18} \frac{\text{kg}}{\text{m}^2}$.
- Boundary conditions: $f_e(t, \|\mathbf{x}\| = L, \mathbf{v}) = 0, \quad \phi(t, \|\mathbf{x}\| = L) = 0$.

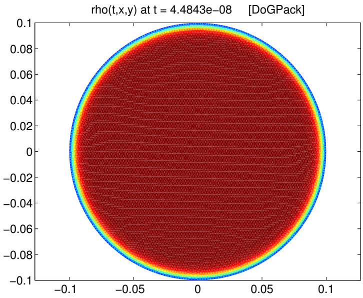


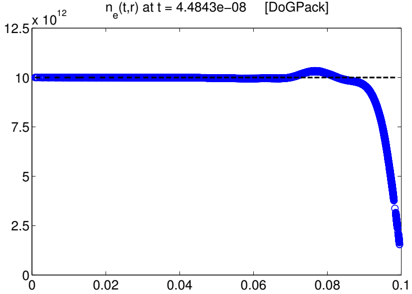


1 eV electrons, $\left(\pi \cdot 0.1^2\right)$ m^2 domain.

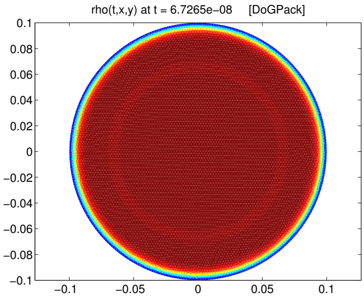
10,000×35×35×15=183,750,000 unknowns

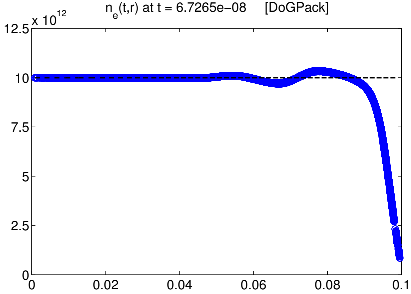
Plasma Sheath Simulation (2D-2V) with HSLDG



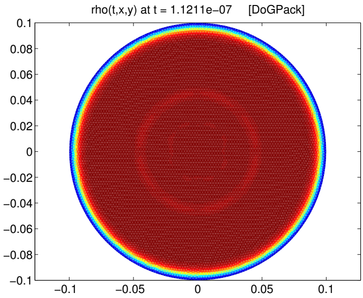


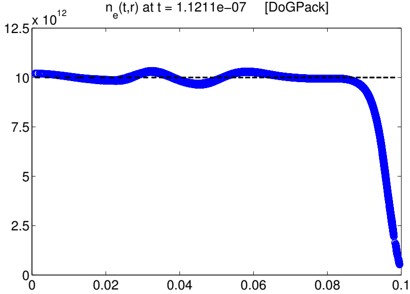
Plasma Sheath Simulation (2D-2V) with HSLDG



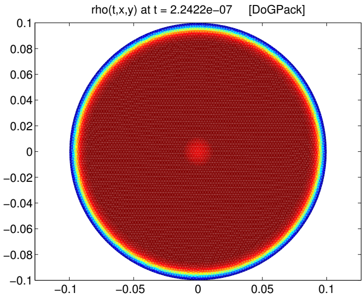


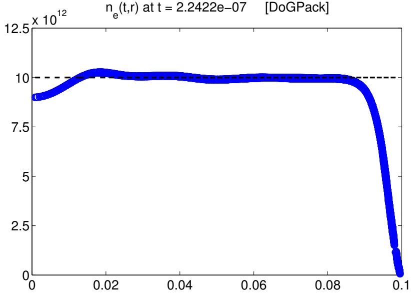
Plasma Sheath Simulation (2D-2V) with HSLDG



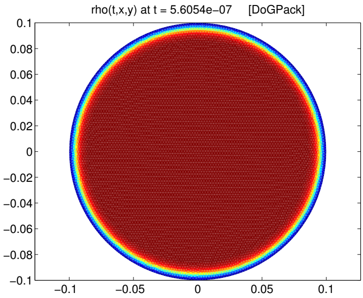


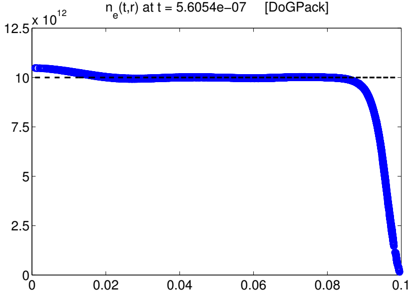
Plasma Sheath Simulation (2D-2V) with HSLDG



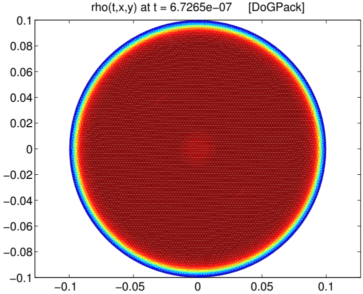


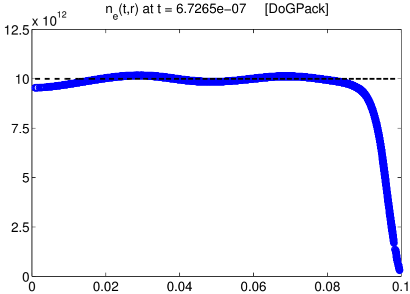
Plasma Sheath Simulation (2D-2V) with HSLDG





Plasma Sheath Simulation (2D-2V) with HSLDG





So whats up with the oscillations?

1D-1V: Matrix sheath (symmetric) - 2nd order (AOFD)

(Loading DoubleSheath_2ndOrder.mp4)

GRID

$N_x = 1024$
$N_v = 512$

TIME STEPPING

$\Delta t = 0.2$
1571 steps
Strang splitting

COURANT NUMBER

$C_x \approx 24.5$
$C_v \approx 18.1$

1D-1V: Matrix sheath (symmetric) - 12th order (AOFD)

(Loading DoubleSheath_12thOrder.mp4)

GRID

$N_x = 1024$
$N_v = 512$

TIME STEPPING

$\Delta t = 0.2$
1571 steps
Strang splitting

COURANT NUMBER

$C_x \approx 24.5$
$C_v \approx 16.4$

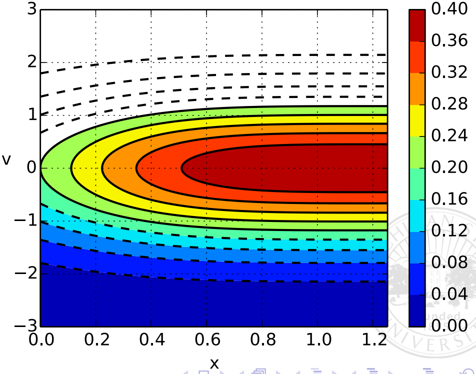
What Do We Know and How Does That Relate to Sheath Theory?

Stationary electron distribution near absorbing wall

- $f(t, x, v)$ is constant along phase-space trajectories (characteristics)
- In steady-state, characteristics are unbroken curves with constant total energy:

$$W_e(x, v) := \frac{1}{2} m_e v^2 - e\phi(x), \qquad W_i(x, v) := \frac{1}{2} m_i v^2 + e\phi(x)$$

- Characteristics symmetric with respect to $v = 0$ line
- Boundary breaks isoenergy curve into two separate characteristics
- See Figure: electrons injected from the right, most energetic reach wall on left ⇒ empty region (white)



Electrons in Boltzmann equilibrium

ASSUMPTIONS:

- Near steady-state, $\partial/\partial t(\cdot) \approx 0 \Rightarrow$ distribution function of total energy
- Electrons Maxwellian at $x = x_0$, and $\phi(x_0) = 0$;
- Negligible phase-space depletion (no electrons reach wall, $\phi_w \to -\infty$).

ELECTRON DISTRIBUTION:

$$f_e(x, v) = \frac{1}{\sqrt{2\pi K T_e}} \exp\left(-\frac{\frac{1}{2} m_e v^2 - e\phi(x)}{K T_e}\right) = f_e(x_0, v) \cdot \exp\left(\frac{e\phi(x)}{K T_e}\right)$$

ELECTRON DENSITY:

$$n_e(x) = n_e(x_0) \cdot \exp\left(\frac{e\phi(x)}{K T_e}\right)$$

Collisionless sheath theory for Boltzmann electrons and cold ions ($T_i \to 0$)

- Poisson's equation: $\dfrac{d^2}{dx^2}\phi(x) = \dfrac{e}{\epsilon_0}\left[n_e(x) - n_i(x)\right]$
- Boltzmann relation for electrons: $n_e(x) = n_{e,0} \cdot \exp\left(\dfrac{e\phi(x)}{KT_e}\right)$
- Conservation of ion number and total energy:

$$\begin{cases} n_i(x)u_i(x) \;=\; n_{i,0}u_{i,0} \\ \dfrac{1}{2}m_i u_i^2(x) + e\phi(x) = \dfrac{1}{2}m_i u_{i,0}^2 \end{cases} \quad\Rightarrow\quad n_i(x) \;=\; n_{i,0}\left(1 - \frac{2e\phi(x)}{m_i u_{i,0}^2}\right)^{-0.5}$$

- Assume **wall** at $x = 0$, **sheath edge** at $x = x_0$:
  - Zero electric field, $\frac{d}{dx}\phi(x_0) = 0$
  - Local charge neutrality, $n_{e,0} = n_{i,0} = n_0$
  - Repelling potential for electrons: $n_i(x) \geq n_e(x)$ for $x \leq x_0$

Collisionless sheath theory for Boltzmann electrons and cold ions ($T_i \to 0$)

- Requirement of repelling potential reduces to
$$\frac{dn_e}{d\phi} \geq \frac{dn_i}{d\phi} \quad \text{for } \phi \leq 0$$
- Taylor expand for small $\phi < 0$ to get **Bohm criterion** [a]:
Ions must enter sheath with velocity larger than **ion acustic speed** c_s
$$u_{i,0} \geq c_s := \sqrt{\frac{KT_e}{m_i}}$$
- Rigorous two-scale theory [b] (sheath+presheath) yields **equality** $u_{i,0} = c_s$
- For general distributions, **generalized Bohm criterion** [c] holds instead:
$$\frac{1}{m_i}\int_{-\infty}^{\infty} \frac{f_i(v)}{v^2} dv = -\frac{1}{m_e}\int_{-\infty}^{\infty} \frac{1}{v}\frac{df_e(v)}{dv} dv$$

[a] D. Bohm. **The characteristics of electrical discharges in magnetic fields**, ed Guthry & Wakerling, McGraw-Hill, p. 77 (1949).

[b] A. Caruso and A. Cavaliere. **The structure of the collisionless plasma-sheath transition**, *Il Nuovo Cimento*, Vol. 26, No. 6, pp. 1389–1404 (1962).

[c] J.E. Allen. **The plasma–sheath boundary: its history and Langmuir's definition of the sheath edge**, *Plasma Sources Science and Technology*, Vol. 18, p. 014004 (2009).

Consider other 1D-1V bounded test cases,
Three Recognized Sheath Test Problems In Literature:

- Sheath at a Floating Wall
  (V. Kolobov, R. Arslanbekov - 2012)
  Stationery Electrons, Drifting Ions
- Source-Collector
  (N. Rizopoulou, A.P.L. Robinson, M. Coppins, and M. Bacharis - 2013)
  Drifting Electrons, Drifting Ions
- Generalized Bohm Criterion
  (J.E. Allen - 2009)
  Stationery Electrons, Generalized Bohm For Ions

Approach: we will go over the problem setup and then results.
HARD PART: Bringing boundary conditions in from infinity.

Sheath Prob 1: Sheath at a floating wall
(stationary electrons and drifting ions)

Sheath Prob 1: Sheath at a floating wall [a,b]

- Perfectly absorbing **wall** at $x = 0$, with floating potential
- **Sheath edge** at $x = L$: Maxw. electrons, drift-Maxw. ions (sonic)
- Accumulated charge $Q(t)$ on wall determines electric field $E(t, 0)$
- Reference potential at sheath edge: $\phi(t, L) = 0 \quad \forall t$
- Steady-state defined by mass ratio $\mu := m_i/m_e$, temperature ratio $\tau := T_i/T_e$, system size L

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial v}\right) f_e(t, x, v) = 0$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{1}{\mu}\frac{\partial \phi}{\partial x}\frac{\partial}{\partial v}\right) f_i(t, x, v) = 0$$

$$\frac{\partial^2}{\partial x^2}\phi(t, x) = \int_{-\infty}^{+\infty} \Big[f_e(t, x, v) - f_i(t, x, v)\Big] dv$$

[a] V. Kolobov, R. Arslanbekov, K. Hara, and I. Boyd. **Collisionless sheath problem for testing Vlasov solvers**, AMVV–2012, 12-15 Nov 2012, East Lansing (MI), USA.

[b] V. Kolobov and R. Arslanbekov. **Towards adaptive kinetic-fluid simulations of weakly ionized plasmas**, *Journal of Computational Physics*, Vol. 231, pp. 839-869 (2012).

Sheath at a floating wall [a,b]

- Surface charge at the wall:

$$Q(t) = \int_0^t \int_{-\infty}^0 v\left[f_e(t',0,v) - f_i(t',0,v)\right] dv\, dt'$$

- Boundary conditions on potential:

$$\frac{\partial}{\partial x}\phi(t,0) = -\frac{1}{2}Q(t), \qquad \phi(t,L) = 0$$

- Boundary conditions on distributions at $x = 0$ (wall):

$$f_e(t,0,v) = f_i(t,0,v) = 0 \qquad \text{if } v > 0$$

[a] V. Kolobov, R. Arslanbekov, K. Hara, and I. Boyd. **Collisionless sheath problem for testing Vlasov solvers**, AMVV–2012, 12-15 Nov 2012, East Lansing (MI), USA.

[b] V. Kolobov and R. Arslanbekov. **Towards adaptive kinetic-fluid simulations of weakly ionized plasmas**, *Journal of Computational Physics*, Vol. 231, pp. 839-869 (2012).

Sheath at a floating wall [a,b]

- Boundary conditions on distributions at $x = L$ (sheath edge):

$$f_e(t,L,v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) \qquad \text{if } v < 0$$

$$f_i(t,L,v) = \frac{1}{\sqrt{2\pi\tau/\mu}} \exp\left(-\frac{(v+c_s)^2}{2\tau/\mu}\right) \qquad \text{if } v < 0$$

- Ion acoustic speed:

$$c_s := \sqrt{\frac{KT_e}{m_i}} = \frac{v_{th,e}}{\sqrt{\mu}} = [\text{non-dimensionalization}] = \frac{1}{\sqrt{\mu}}$$

- System length, mass ratio, temperature ratio:

$$L = 20 \qquad \mu \approx 3671.5\ (^2\text{H} - \text{Deuterium}) \qquad \tau = 1/30$$

[a] V. Kolobov, R. Arslanbekov, K. Hara, and I. Boyd. **Collisionless sheath problem for testing Vlasov solvers**, AMVV–2012, 12-15 Nov 2012, East Lansing (MI), USA.

[b] V. Kolobov and R. Arslanbekov. **Towards adaptive kinetic-fluid simulations of weakly ionized plasmas**, *Journal of Computational Physics*, Vol. 231, pp. 839-869 (2012).

Floating wall: ion distribution vs. time

(Loading Distribution_Ions_L=20_mu=3671.mp4)

FINAL TIME

$t_{\text{end}} = 2000 \times 2\pi$

SOLVER

6th-order CS

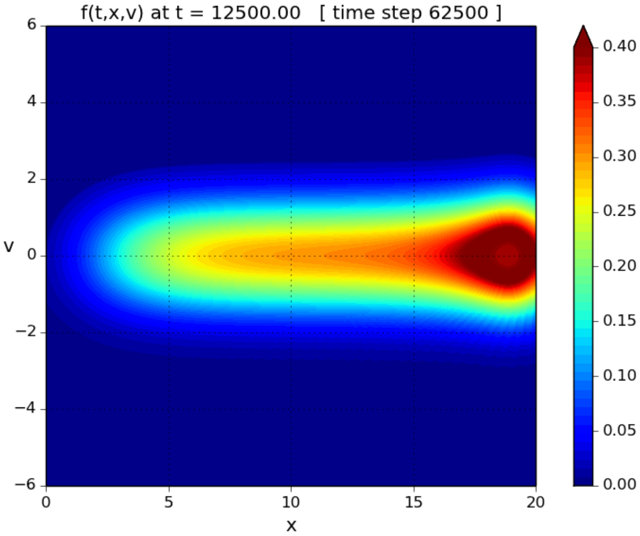
GRID

$N_x = 240$
$N_v = 300$

TIME STEPPING

$\Delta t = 0.2$
62832 steps
Strang splitting

Floating wall: final electron distribution



FINAL TIME

$t_{\text{end}} = 2000 \times 2\pi$

SOLVER

6th-order CS

GRID

$N_x = 240$

$N_v = 300$

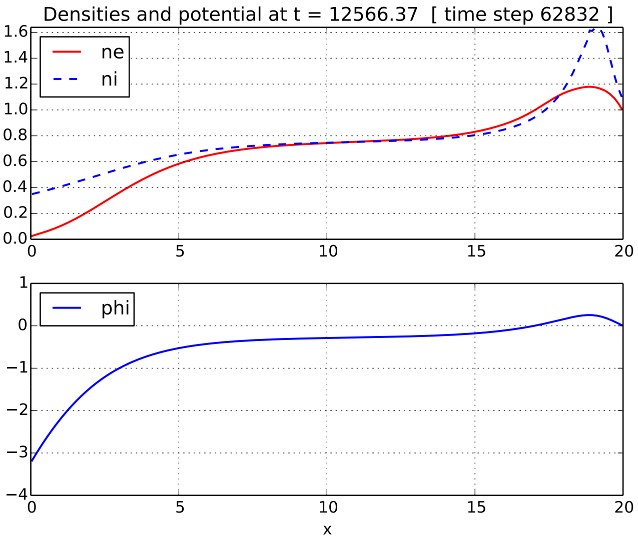
TIME STEPPING

$\Delta t = 0.2$

62832 steps

Strang splitting

Floating wall: sheath (left) and virtual cathode (right)



FINAL TIME

$t_{\text{end}} = 2000 \times 2\pi$

SOLVER

6th-order CS

GRID

$N_x = 240$

$N_v = 300$

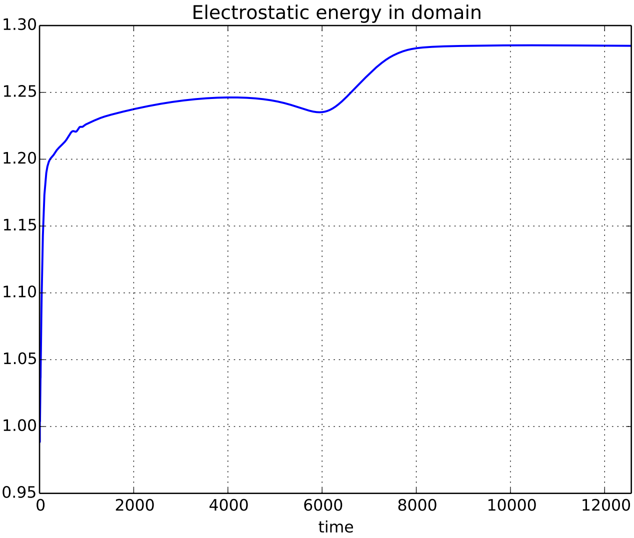
TIME STEPPING

$\Delta t = 0.2$

62832 steps

Strang splitting

Floating wall: relaxation to steady-state



Final time
$t_{\text{end}} = 2000 \times 2\pi$

Solver
6th-order CS

Grid
$N_x = 240$
$N_v = 300$

Time stepping
$\Delta t = 0.2$
62832 steps
Strang splitting

Floating wall, $L = 30$: ion distribution vs. time

(Loading Distribution_Ions_L=30_mu=3671.mp4)

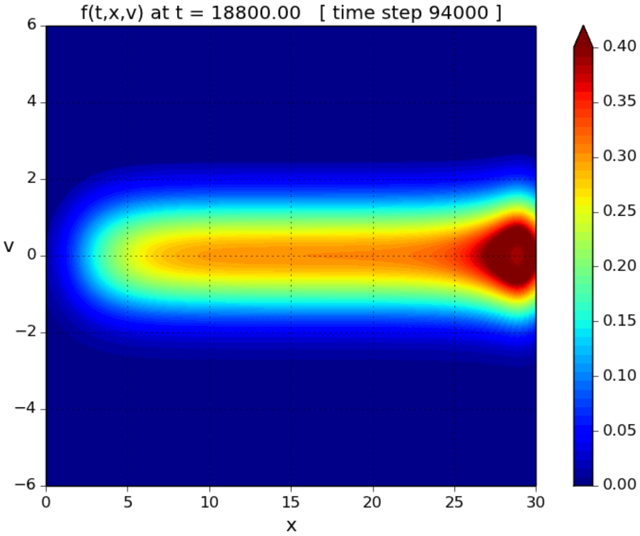
FINAL TIME
$t_{\text{end}} = 3000 \times 2\pi$

SOLVER
6th-order CS

GRID
$N_x = 360$
$N_v = 300$

TIME STEPPING
$\Delta t = 0.2$
94248 steps
Strang splitting

Floating wall, $L = 30$: final electron distribution



FINAL TIME

$t_{\text{end}} = 3000 \times 2\pi$

SOLVER

6th-order CS

GRID

$N_x = 360$

$N_v = 300$

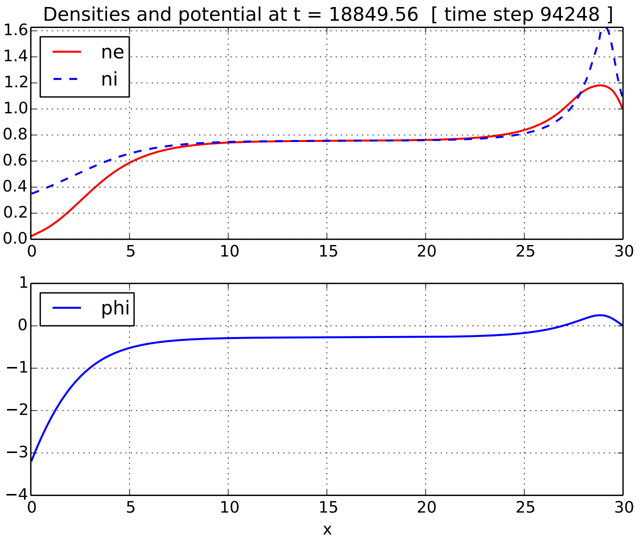
TIME STEPPING

$\Delta t = 0.2$

94248 steps

Strang splitting

Floating wall, $L = 30$: sheath and virtual cathode



FINAL TIME

$t_{\text{end}} = 3000 \times 2\pi$

SOLVER

6th-order CS

GRID

$N_x = 360$

$N_v = 300$

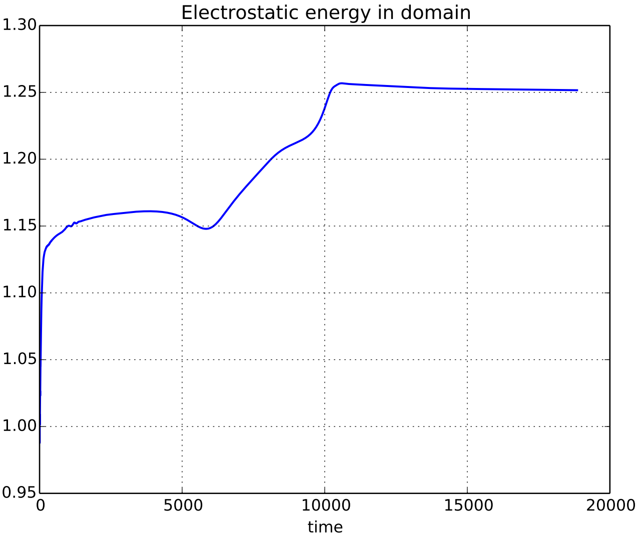
TIME STEPPING

$\Delta t = 0.2$

94248 steps

Strang splitting

Floating wall, $L = 30$: relaxation to steady-state



FINAL TIME

$t_{\text{end}} = 3000 \times 2\pi$

SOLVER

6th-order CS

GRID

$N_x = 360$

$N_v = 300$

TIME STEPPING

$\Delta t = 0.2$

94248 steps

Strang splitting

Sheath Prob 2: The source-collector problem
(drifting electrons and drifting ions)

Sheath Prob 2: The source-collector problem [a]

- Floating electrode at $x = 0$, perfectly absorbing (**collector**)
- Incoming electrons and ions at $x = L$, drift-Maxwellian (**source**)
- Accumulated charge $Q(t)$ on electrode defines electric field $E(t, 0)$
- Reference potential at source: $\phi(t, L) = 0 \quad \forall t$
- Steady-state defined by mass ratio $\mu := m_i/m_e$, temperature ratio $\tau := T_i/T_e$, drift velocity v_d (in units of ion acoustic speed c_s)

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial v}\right) f_e(t, x, v) = 0$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{1}{\mu}\frac{\partial \phi}{\partial x}\frac{\partial}{\partial v}\right) f_i(t, x, v) = 0$$

$$\frac{\partial^2}{\partial x^2}\phi(t, x) = \int_{-\infty}^{+\infty} \Big[f_e(t, x, v) - f_i(t, x, v)\Big] dv$$

[a] N. Rizopoulou, A.P.L. Robinson, M. Coppins, and M. Bacharis. **A kinetic study of the source-collector sheath system in a drifting plasma**, *Plasma Sources Science and Technology*, Vol. 22, p. 035003 (2013).

The source-collector problem [a]

- Surface charge at collector:

$$Q(t) = \int_0^t \int_{-\infty}^0 v \left[f_e(t', 0, v) - f_i(t', 0, v) \right] dv\, dt'$$

- Boundary conditions on potential:

$$\frac{\partial}{\partial x}\phi(t,0) = -\frac{1}{2}Q(t) \qquad \phi(t,L) = 0$$

- Boundary conditions on distributions (collector):

$$f_e(t,0,v) \; = \; f_i(t,0,v) \; = \; 0 \qquad \text{if } v > 0$$

[a] N. Rizopoulou, A.P.L. Robinson, M. Coppins, and M. Bacharis. **A kinetic study of the source-collector sheath system in a drifting plasma**, *Plasma Sources Science and Technology*, Vol. 22, p. 035003 (2013).

The source-collector problem [a]

- Boundary conditions on distributions at $x = L$ (source):

$$f_e(t, L, v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(v - v_d)^2}{2}\right) \qquad \text{if } v < 0$$

$$f_i(t, L, v) = \frac{1}{\sqrt{2\pi\tau/\mu}} \exp\left(-\frac{(v - v_d)^2}{2\tau/\mu}\right) \qquad \text{if } v < 0$$

- System length, mass and temperature ratios, drift velocities:
  - $L \approx 53.35$
  - $\tau = 1$ (thermal plasma)
  - $\mu \approx 1836$ (Hydrogen)
  - $v_d \in \{0, c_s, 3c_s, 4c_s\}$

- Ion acoustic speed:

$$c_s := \sqrt{\frac{KT_e}{m_i}} = \frac{v_{th,e}}{\sqrt{\mu}} = [\text{non-dimensionalization}] = \frac{1}{\sqrt{\mu}}$$

[a] N. Rizopoulou, A.P.L. Robinson, M. Coppins, and M. Bacharis. **A kinetic study of the source-collector sheath system in a drifting plasma**, *Plasma Sources Science and Technology*, Vol. 22, p. 035003 (2013).

$v_d = 0$: ion distribution vs. time

(Loading Distribution_Ions_tau=1_vd=0.mp4)

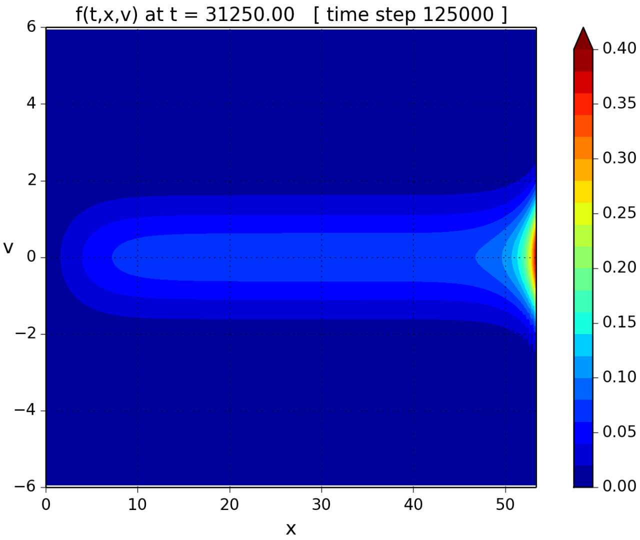
FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = 0$: final electron distribution



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

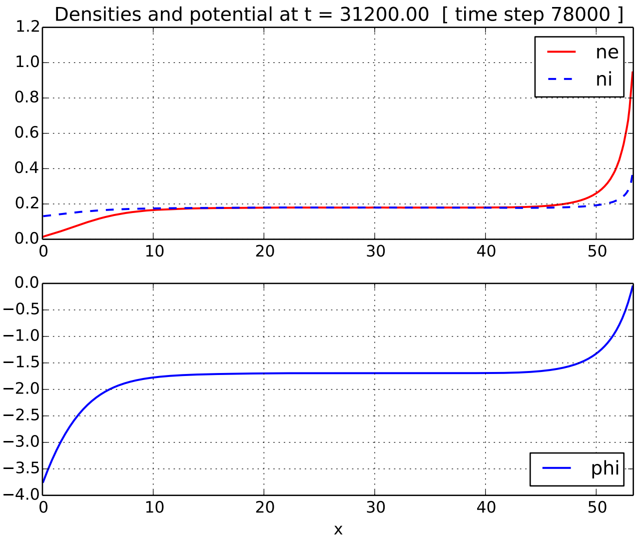
TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = 0$: collector and source sheaths



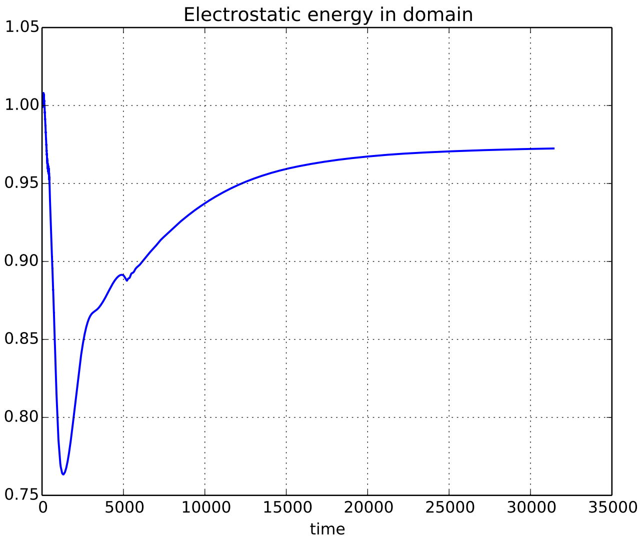
FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = 0$: relaxation to steady-state



FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = c_s$: ion distribution vs. time

(Loading Distribution_Ions_tau=1_vd=1.mp4)

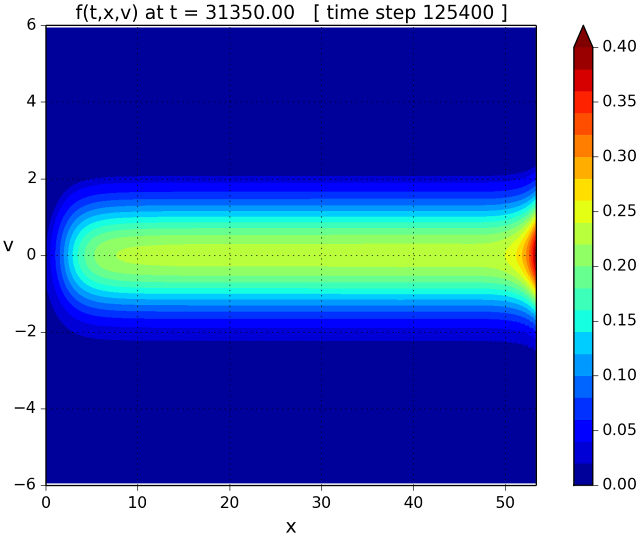
FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = c_s$: final electron distribution



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

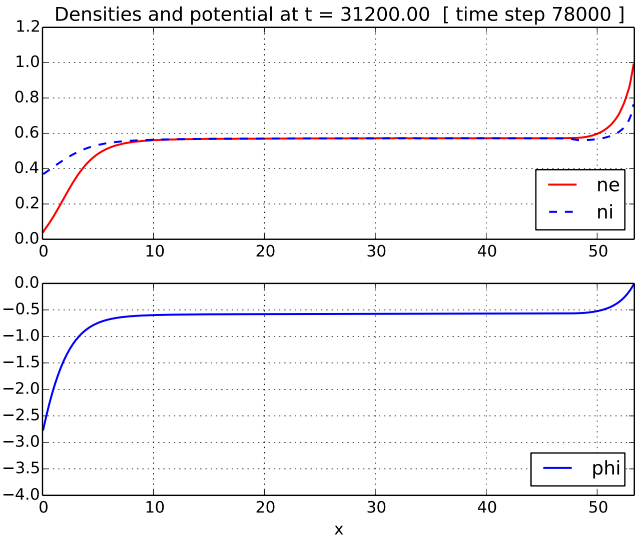
TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = c_s$: collector and source sheaths



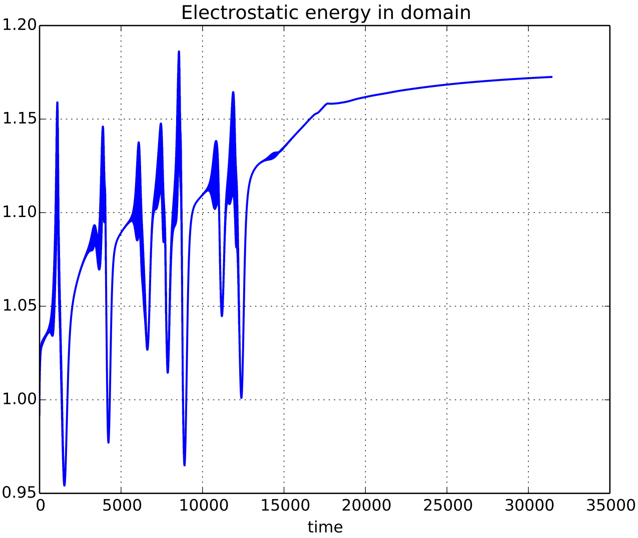
Final time
$t_{\text{end}} = 5000 \times 2\pi$

Solver
4th-order CS

Grid
$N_x = 400$
$N_v = 100$

Time stepping
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = c_s$: relaxation to steady-state



FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = 3c_s$: ion distribution vs. time

(Loading Distribution_Ions_tau=1_vd=3.mp4)

FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

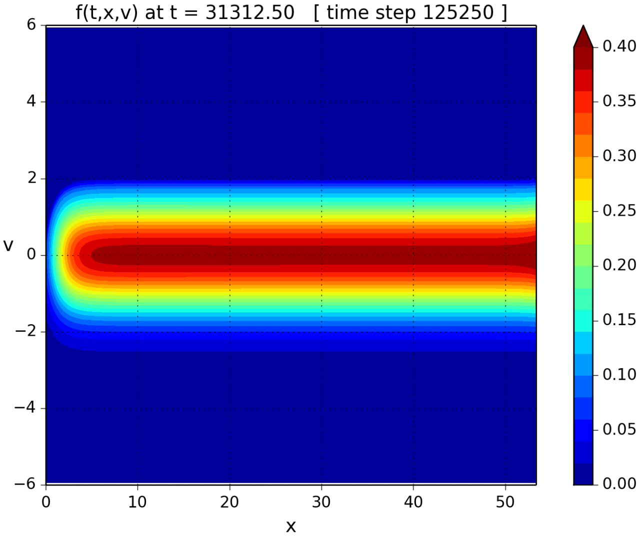
TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = 3c_s$: final electron distribution



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

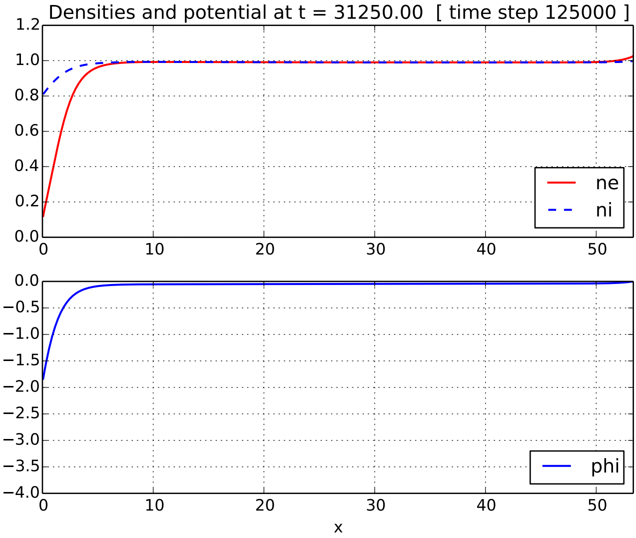
TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = 3c_s$: collector and source sheaths



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

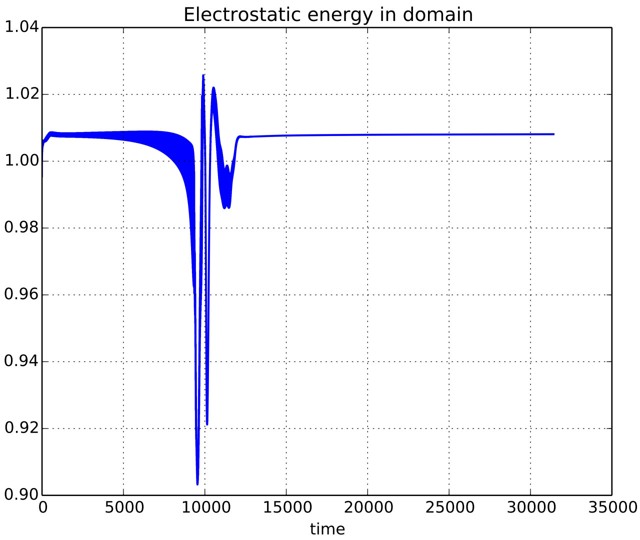
TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = 3c_s$: relaxation to steady-state



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = 4c_s$: ion distribution vs. time

(Loading Distribution_Ions_tau=1_vd=4.mp4)

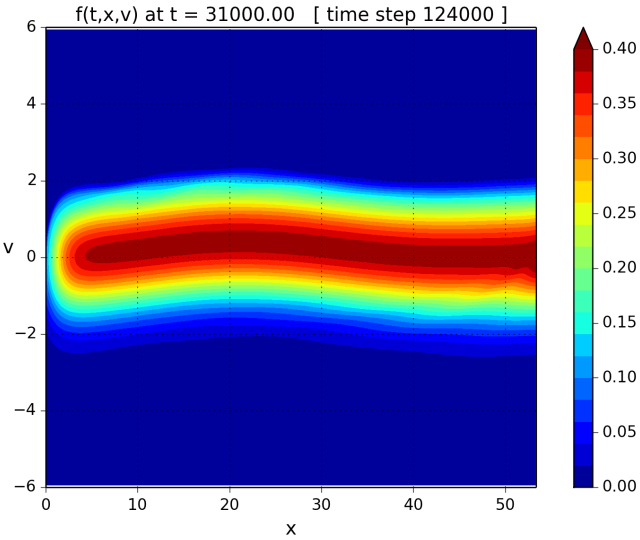
FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = 4c_s$: final electron distribution



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

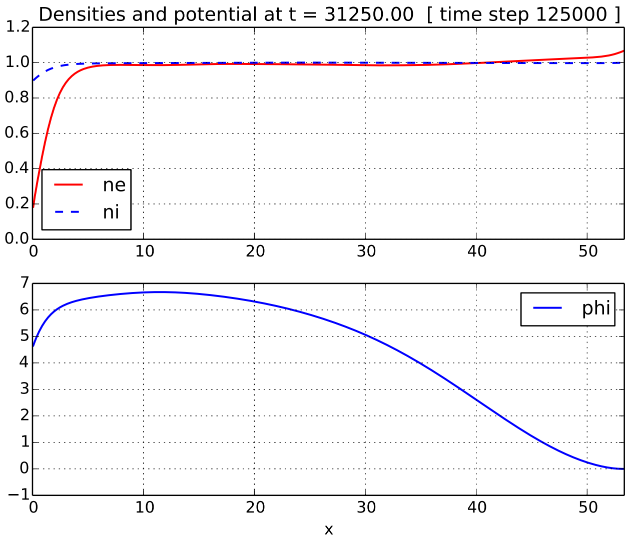
TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

$v_d = 4c_s$: collector and source sheaths



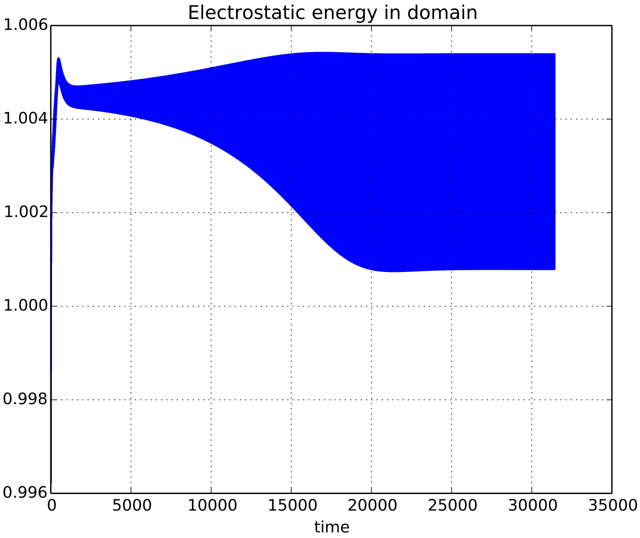
FINAL TIME
$t_{\text{end}} = 5000 \times 2\pi$

SOLVER
4th-order CS

GRID
$N_x = 400$
$N_v = 100$

TIME STEPPING
$\Delta t = 0.25$
125664 steps
Strang splitting

$v_d = 4c_s$: relaxation to steady-state??



FINAL TIME

$t_{\text{end}} = 5000 \times 2\pi$

SOLVER

4th-order CS

GRID

$N_x = 400$

$N_v = 100$

TIME STEPPING

$\Delta t = 0.25$

125664 steps

Strang splitting

Sheath Prob 3: Distributions satisfying the generalized Bohm criterion (using our observations about cutoff distributions)

Sheath Prob 3: Distributions satisfying the generalized Bohm criterion

CONSTRAINTS AT SHEATH EDGE ($x = L$)

- Charge neutrality:

$$\int_{-\infty}^{+\infty} f_e(v)\,dv = \int_{-\infty}^{0} f_i(v)\,dv = 1$$

- Zero net current:

$$\int_{-\infty}^{+\infty} v\,f_e(v)\,dv = \int_{-\infty}^{0} v\,f_i(v)\,dv = u_0$$

- Sonic ions = Generalized Bohm criterion [a]:

$$-\int_{-\infty}^{+\infty} \frac{1}{v}\frac{df_e(v)}{dv}dv = \frac{1}{\mu}\int_{-\infty}^{0} \frac{f_i(v)}{v^2}dv$$

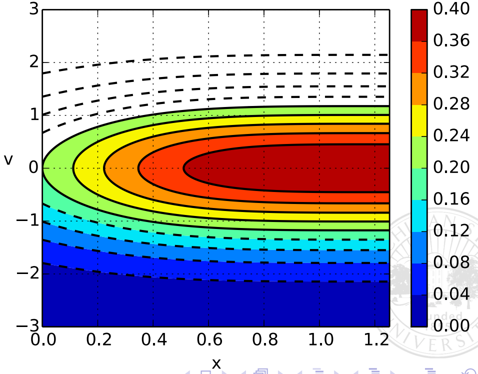
[a] J.E. ALLEN. **The plasma–sheath boundary: its history and Langmuir's definition of the sheath edge**, *Plasma Sources Science and Technology*, Vol. 18, p. 014004 (2009).

Recall observation about stationary electron near wall

- $f(t, x, v)$ is constant along phase-space trajectories (characteristics)
- In steady-state, characteristics are unbroken curves with constant total energy:

$$W_e(x, v) := \frac{1}{2}m_e v^2 - e\phi(x), \qquad W_i(x, v) := \frac{1}{2}m_i v^2 + e\phi(x)$$

- Characteristics symmetric with respect to $v = 0$ line
- Boundary breaks isoenergy curve into two separate characteristics
- See Figure: electrons injected from the right, most energetic reach wall on left $\Rightarrow$ empty region (white)



Distributions satisfying the generalized Bohm criterion

ELECTRONS

- **Truncated Maxwellian** distribution at sheath edge

$$f_e(v) = \frac{A_e}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) \cdot H\left(v_{\max,e} - v\right)$$

- Obtain $A_e > 0$ and $v_{\max,e} > 0$ from density and mean velocity

$$1 = \int_{-\infty}^{+\infty} f_e(v)\, dv = A_e \cdot \mathrm{normcdf}\left(v_{\max,e}\right)$$

$$u_0 = \int_{-\infty}^{+\infty} v\, f_e(v)\, dv = -\frac{A_e}{\sqrt{2\pi}} \exp\left(-\frac{v_{\max,e}^2}{2}\right)$$

- LHS of generalized Bohm criterion yields

$$-\int_{-\infty}^{+\infty} \frac{1}{v} \frac{df_e(v)}{dv} dv = A_e \cdot \mathrm{normcdf}\left(v_{\max,e}\right) \equiv 1$$

Distributions satisfying the generalized Bohm criterion

IONS

- Drift-Maxwellian distribution (sonic), accelerated through potential drop ψ

$$f_i(v) = \frac{A_i}{\sqrt{2\pi\tau/\mu}} \exp\left(-\frac{\left(\sqrt{v^2 - v_{\max,i}^2} + c_s\right)^2}{2\tau/\mu}\right) \cdot H\left(v_{\max,i} - v\right)$$

$$v_{\max,i} = -\sqrt{2\psi/\mu}$$

- $A_i > 0$ and $v_{\max,i} < 0$ obtained from continuity and Bohm criterion

$$\int_{-\infty}^{v_{\max,i}} f_i(v)\, dv = 1 \qquad \frac{1}{\mu}\int_{-\infty}^{v_{\max,i}} \frac{f_i(v)}{v^2} dv = 1$$

- Given $f_i(v)$, calculated $u_0 < 0$ and pass it to electrons (note $|u_0| > c_s$)

$$u_0 := \int_{-\infty}^{v_{\max,i}} v\, f_i(v)\, dv$$

Floating wall, $\mu = 100$, uncorrected BCs

(Loading Distribution_Ions_L=30_mu=100.mp4)

FINAL TIME

$t_{\text{end}} = 500 \times 2\pi$

SOLVER

6th-order CS

GRID

$N_x = 360$
$N_v = 300$

TIME STEPPING

$\Delta t = 0.2$
15708 steps
Strang splitting

Floating wall, $\mu = 100$, corrected electron BCs

(Loading Distribution_Ions_L=30_mu=100.mp4)

FINAL TIME
$t_{\mathrm{end}} = 500 \times 2\pi$

SOLVER
6th-order CS

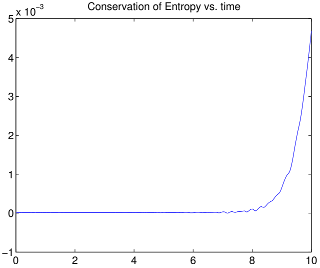
GRID
$N_x = 360$
$N_v = 300$

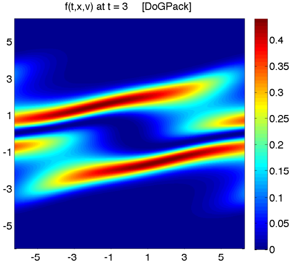
TIME STEPPING
$\Delta t = 0.2$
15708 steps
Strang splitting

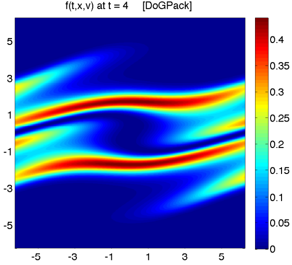
Conclusions

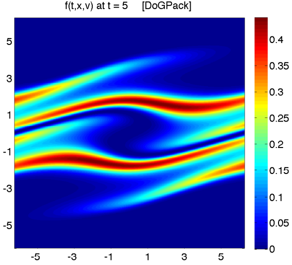
- Bounded domains are challenging for Vlasov Codes:
    - Very strong electric field in narrow boundary layer (plasma sheath)
    - Multiple species $\Rightarrow$ large time-scale separation
    - Long and complex transient before reaching steady-state solution
- Matrix sheath problem (electrons only): no well-defined steady-state
- Stationary sheath requires supersonic ions (generalized Bohm criterion)
- If boundary conditions not correct, plasma 'adjusts itself':
    - Low-energy ions either repelled or accelerated
    - Kolobov's setup $\Rightarrow$ long-time **virtual cathode** formation
    - Drift-Maxwellian electrons $\Rightarrow$ long-time **source sheath** formation
- Boundary conditions can be improved (WIP)

One Final Thought
Is Vlasov the right model?









Acknowledgments

This work was partially supported by the following agencies:

- MICHIGAN STATE UNIVERSITY FOUNDATION
  SPG-RG100059
- AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFOSR)
  FA9550-11-1-0281, FA9550-12-1-0343, FA9550-12-1-0455
- NATIONAL SCIENCE FOUNDATION (NSF)
  DMS-1115709