High Order Weighted Essentially Non-Oscillatory Adaptive Mesh Refinement Method for Ideal Magnetohydrodynamic Equations

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Abstract

The new Weighted Essentially Non-Oscillatory (WENO) Adaptive Mesh Refinement (AMR) algorithm [2] is used to simulate the ideal magnetohydrodynamic (MHD) Equations. The magnetic potential advection constrained transport method (MPACT) [1] is used to satisfy the divergence-free constraint of magnetic field. 1D benchmark problem Brio-Wu shock tube is presented in the AMR framework. We expect our algorithm is robust, essentially non-oscillatory and suitable for resolving solution structures

1. Ideal MHD Equations

The ideal MHD Equations, in conservation form, can be written as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \mathcal{E}\\ \mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\otimes\mathbf{u}+(p+\frac{1}{2}B^2)\mathbf{I}-\mathbf{B}\otimes\mathbf{B}\\ \mathbf{u}(\mathcal{E}+p+\frac{1}{2}B^2)-\mathbf{B}(\mathbf{u}\cdot\mathbf{B})\\ \mathbf{u}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{u}\end{bmatrix}=0$$

with divergence-free constraint

$$\nabla\cdot\mathbf{B}=0,$$

where

$$\mathcal{E}=\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+\frac{B^2}{2}$$

Assumption:

1. Single-fluid model of a plasma
2. Quasi-neutrality
3. Slow characteristic velocity of the phenomenon
4. Long time scale compared to electron and ion cyclotron periods
5. Perfect conductor due to little resisitivity

Applications:
Astrophysical jets, Solar tachocline, etc.

2. Seven-Waves and Eight-Waves Eigensystem

Solving 1D ideal MHD Equations

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=0$$

by WENO reconstruction needs the eigenvaules and eigenvectors in the conservative variable space.
The seven-wave system has the well-ordered eigienvaules:

$$\lambda_{1,7}=u^1\mp c_f,\quad \lambda_{2,6}=u^1\mp c_a,\quad \lambda_{3,5}=u^1\mp c_s,\quad \lambda_4=u^1,$$

where

$$U=(\rho,\rho u^1,\rho u^2,\rho u^3,\mathcal{E},B^2,B^3)$$

$$a=\sqrt{\gamma p/\rho},\quad b=||\mathbf{B}||^2/\rho,\quad (b^1,b^2,b^3)=(B^1,B^2,B^3)/\sqrt{\rho}$$

$$c_a=\sqrt{(B^1)^2/\rho},\ c_{f,s}=\left\{\frac{1}{2}\left[a^2+b^2\pm\sqrt{(a^2+b^2)^2-4a^2(b^1)^2}\right]\right\}^{\frac{1}{2}}$$

The eight-wave system considered all the eight-components together, inserting one more eigenvaule,

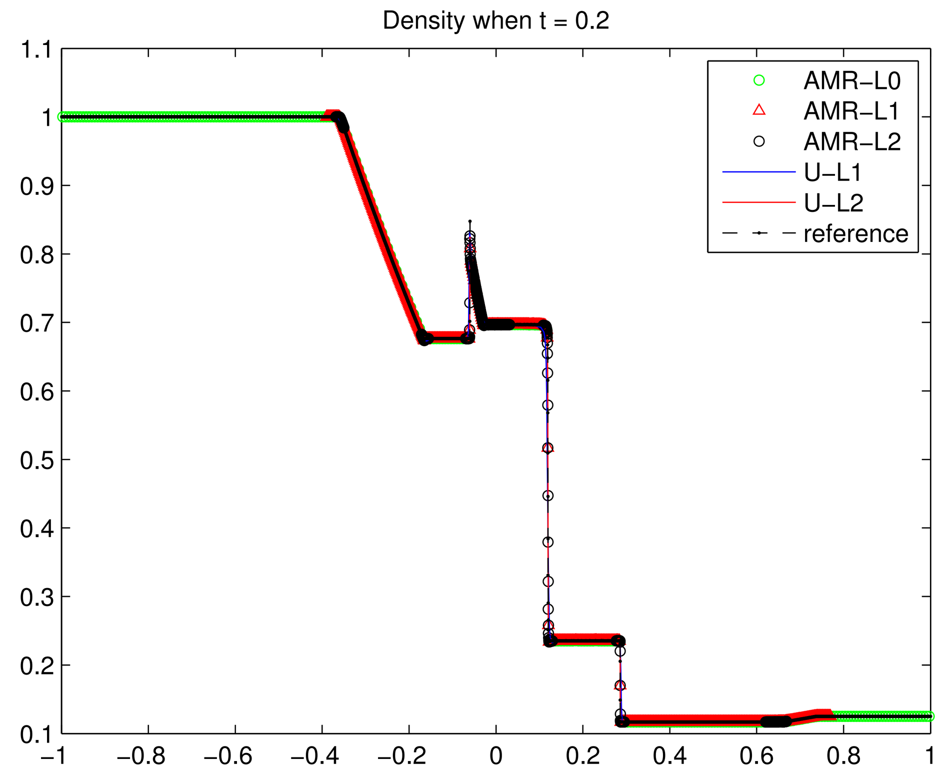
$$\lambda_5=u^1$$

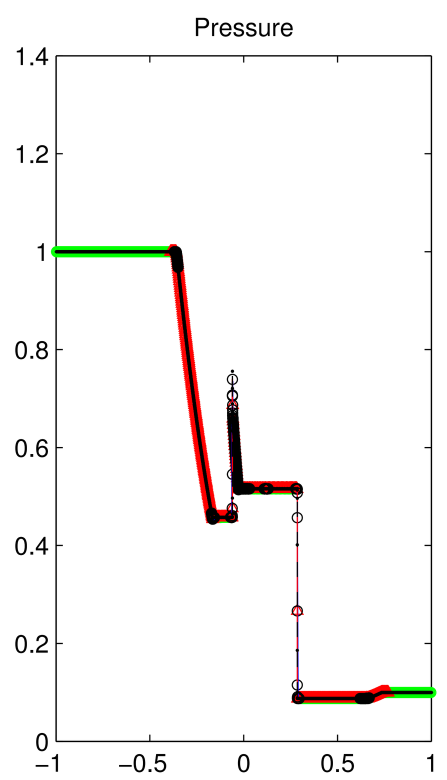
In our work, both eigensystems are implemented. In 1D case, there is almost no difference in solutions. Thus, in the following discussion, only the solution of eight-wave solver is presented. However, it has been observed that there will be significant difference in high dimension cases.

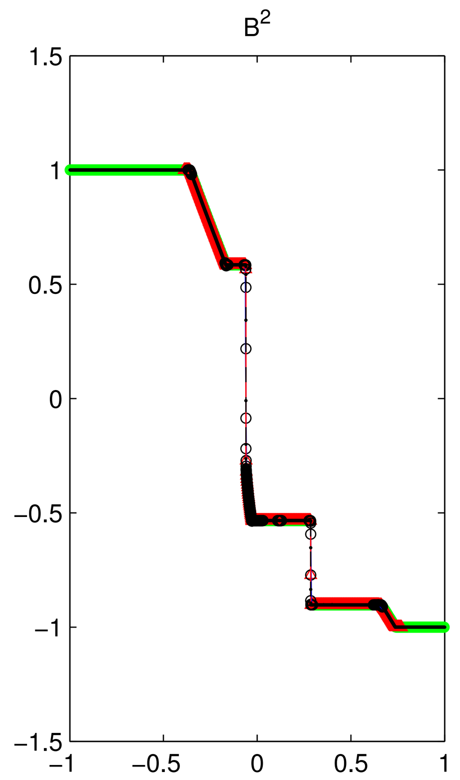
3. 1D Brio Wu Shock Tube

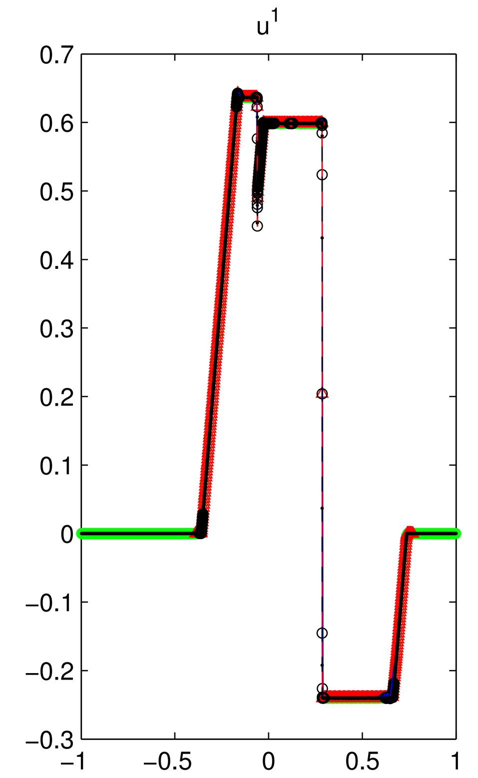
1D Riemann problem with the initial condition:

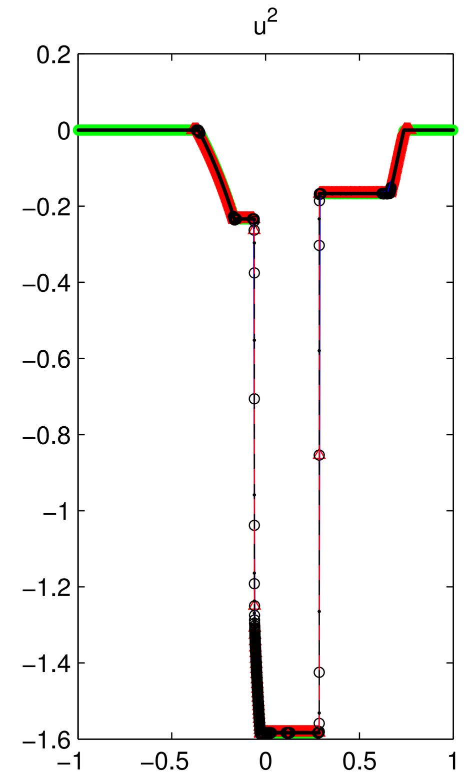
$$(\rho,u^1,u^2,u^3,B^1,B^2,B^3,p)=\begin{cases}(0.125,0,0,0,0.75,-1.0,0,0.1), & 0<x<1\\ (1.0,0,0,0,0.75,1.0,0,1.0), & -1<x\le 0\end{cases}$$

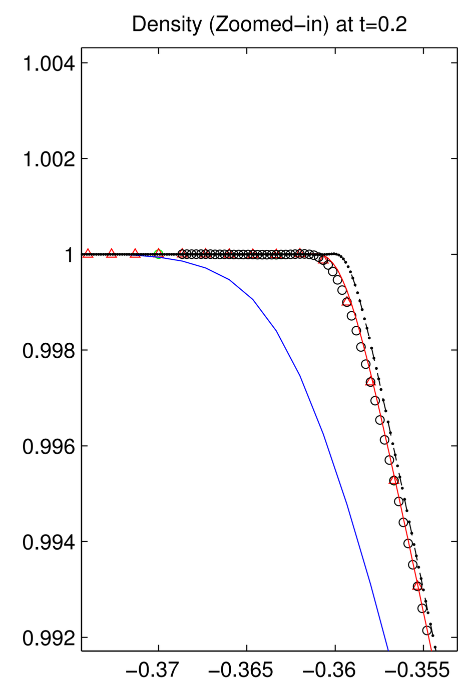


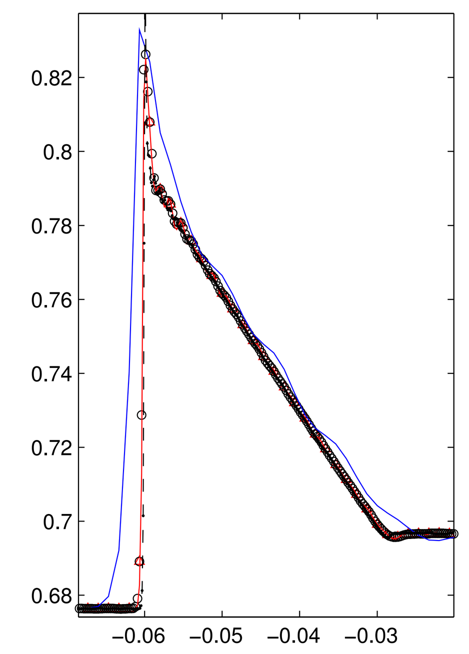


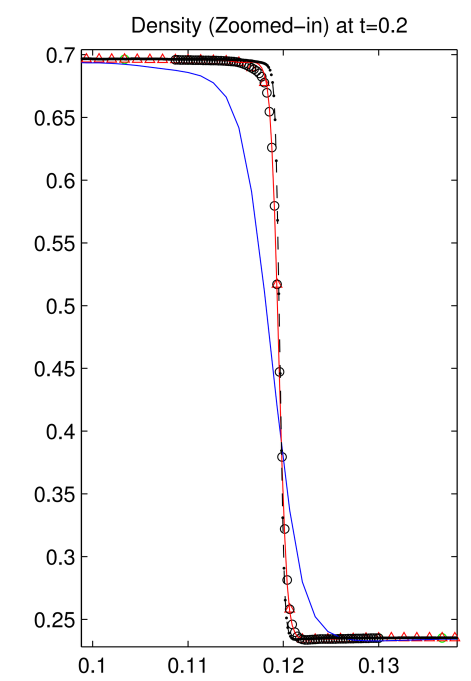


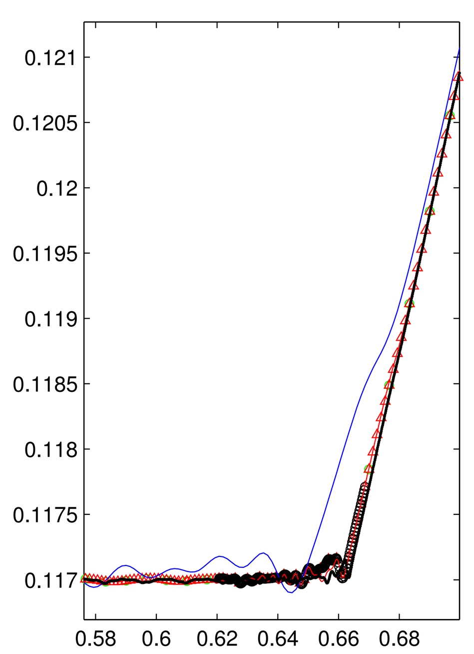












Some remarks:

1. The AMR solution is solved by AMR-WENO with local Lax-Friedrichs flux splitting used.
2. U-L1 denotes the solution of uniform mesh of the same step size as Level 1 mesh. U-L2 has a similar meaning. The reference solution is solved on a fine mesh.
3. The AMR solution matches very well with U-L2.
4. The solutions of finer step sizes show fewer small oscillations. This illustrates the efficiency of WENO in this problem.

4. High Dimension Cases and MPACT

The divergence-free constraint has to been treated specifically in high dimension case. In our work, MPACT [1] is used to achieve an essentially non-oscillatory and divergence-free $\mathbf{B}$ field.
The vector potential of $\mathbf{B}$ field are evolved by

$$\mathbf{A}_t+(\nabla\times\mathbf{A})\times\mathbf{u}=-\nabla\psi,$$

In 2D case, only the 3rd components of $\mathbf{A}$ is needed

$$A^3_t+u^1A^3_x+u^2A^3_y=0,$$

while in 3D case, the weakly hyperbolic system has to be solved

$$\begin{bmatrix}A^1\\A^2\\A^3\end{bmatrix}_t+\begin{bmatrix}0&-u^2&-u^3\\0&u^1&0\\0&0&u^1\end{bmatrix}\begin{bmatrix}A^1\\A^2\\A^3\end{bmatrix}_x+\begin{bmatrix}u^2&0&0\\-u^1&0&-u^3\\0&0&u^2\end{bmatrix}\begin{bmatrix}A^1\\A^2\\A^3\end{bmatrix}_y$$
$$+\begin{bmatrix}u^3&0&0\\0&u^3&0\\-u^1&-u^2&0\end{bmatrix}\begin{bmatrix}A^1\\A^2\\A^3\end{bmatrix}_z=0$$

5. Future Directions

1. Implement 2D/3D Benchmark problems such as Orszag-Tang vortex and Cloud-shock interaction
2. Compare the solutions with those solved by Finite Volume method in [1]

References

[1] Rossmanith, J.A., “*An unstaggered, high-resolution constrained transport method for magnetohydrodynamic flows*”, SIAM J. Sci. Comp., 28 (2006), pp. 1766–1797

[2] Shen, C., Qiu, J.M. and Christlieb, A. “*Adaptive mesh refinement based on high order finite difference WENO scheme for multi-scale simulations*”, J. Comput. Phys., 230 (2011), pp. 3780–3802

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