Finite Difference Weighted Essentially Non-Oscillatory Schemes with Constrained Transport for Ideal MHD

Qi Tang

Department of Mathematics
Michigan State University

Collaborator:
Andrew Christlieb – MSU
James Rossmanith – ISU

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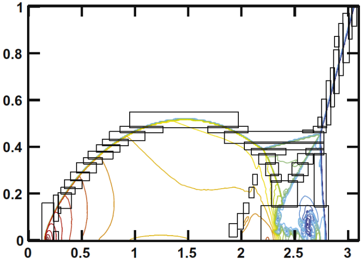
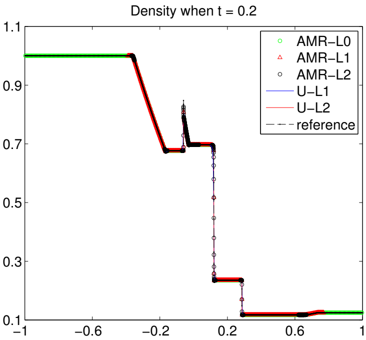
Contents

Motivation

- Adaptive mesh refinement for plasma simulation
- High-order and efficient solver (5^{th}-order FD-WENO)





[C.Shen, J.-M.Qiu, & Christlieb, 2011]

[Tang, etc, 2012]

Ideal MHD equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \mathcal{E}\\ \mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+(p+\frac{1}{2}||\mathbf{B}||^2)\mathbf{I}-\mathbf{B}\mathbf{B}\\ \mathbf{u}(\mathcal{E}+p+\frac{1}{2}||\mathbf{B}||^2)-\mathbf{B}(\mathbf{u}\cdot\mathbf{B})\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\end{bmatrix}=0$$

$$\nabla\cdot\mathbf{B}=0$$

where ρ, $\rho\mathbf{u}$ and $\mathcal{E}$ is the total mass, momentum and energy density of the plasma system, $\mathbf{B}$ is the magnetic field and p is the hydrodynamic pressure. The total energy density is given by

$$\mathcal{E}=\frac{\rho\|\mathbf{u}\|^2}{2}+\frac{p}{\gamma-1}+\frac{\|\mathbf{B}\|^2}{2}$$

$\gamma=5/3$ is the ideal gas constant.

Hyperbolic Conservation Laws

The MHD equation forms a system of *hyperbolic conservation laws*

$$q_{,t} + \nabla \cdot \mathbf{F}(q) = 0,$$

where $q = (\rho, \rho\mathbf{u}, \mathcal{E}, \mathbf{B})$ are the conserved variables and $\mathbf{F}$ is the flux tensor.

Numerical schemes for *HCL*

- Finite Difference (Lax-Friedrichs, artificial viscosity, FD-ENO/WENO)
- Finite Volume (Godunov schemes with Riemann solvers, HLL, HLLC, Roe, etc; MUSCL scheme; Lax-Wendroff type schemes; FV-ENO/WENO)
- Finite Element
- Discontinuous Galerkin
- and many more

High-order FD-WENO scheme for HCL

Consider an 1D scalar HCL

$$q_{,t} + f(u)_{,x} = 0$$

FD-WENO solve this system by:

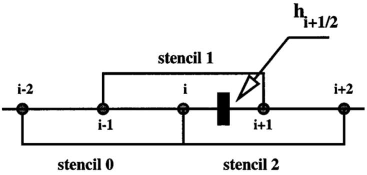
$$\frac{dq_i(t)}{dt} = -\frac{1}{\Delta x}(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}})$$

with numerical flux $\hat{f}_{i+\frac{1}{2}}$,

$$\hat{f}_{i+\frac{1}{2}} = \Phi_{\text{WENO}}(f_{i-p}, \dots, f_{i+q})$$

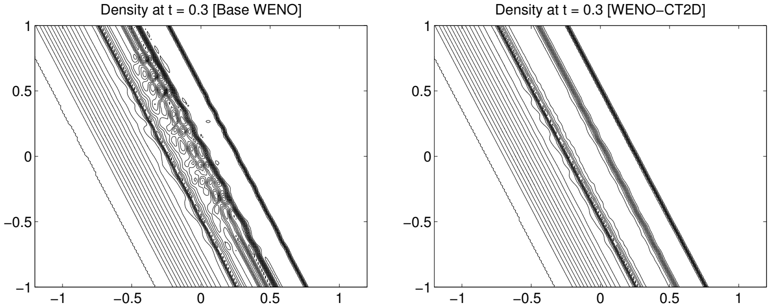
E.g., **WENO5** spatial discretization has the idea:

- If q is **SMOOTH**, becomes a **5th-order central scheme**.
- If **NON-SMOOTH**, close to a **3rd-order Upwinding scheme** $\Rightarrow$ **Essentially No Oscillations**



The role of $\nabla \cdot \mathbf{B} = 0$

Numerical solutions for ideal MHD need to satisfy the divergence free condition or at least control the divergence error on $\mathbf{B}$ in some discrete sense.



No $\nabla \cdot \mathbf{B}$ control vs. $\nabla \cdot \mathbf{B}$ control

[Christlieb, Rossmanith & Tang, 2013]

Approaches to control $\nabla \cdot \mathbf{B}$

Different approaches to control $\nabla \cdot \mathbf{B}$:

- Projection method [Tóth, 2000], [Balsara & Kim, 2004], ...
- 8-wave formulation [Powell, 1994], [Powell, et al, 1999]
- Divergence cleaning approach [Dedner, et al, 1998]
- **Constrained transport methods**
  Staggered: [Dai & Woodward 1998], [Ryu et al, 1998], [Balsara & Spicer, 1999], [De Sterck, 2001], [Balsara, 2004], [Londrillo & Del Zanna 2004], ...

  Unstaggered: [Tóth, 2000], [Fey & Torrilhon, 2003], [Rossmanith, 2006], [Helzel, et al, 2011, 2013] ...

Constrained Transport Method

Introduce the *vector potential* **A**

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

So the evolution equation of **B**

$$\mathbf{B}_{,t} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$$

becomes

$$\begin{aligned} &\nabla \times (\mathbf{A}_{,t} + (\nabla \times \mathbf{A}) \times \mathbf{u}) = 0 \\ &\Rightarrow \mathbf{A}_t + (\nabla \times \mathbf{A}) \times \mathbf{u} = -\nabla \psi \end{aligned}$$

Introducing the Weyl gauge, i.e., setting $\psi \equiv 0$,

$$\mathbf{A}_{,t} + (\nabla \times \mathbf{A}) \times \mathbf{u} = 0$$

In our schemes, **A** and **B** are point values on the same mesh.

See e.g. [C.Helzel, Rossmanith & B.Taetz, 2011] for different choices of **gauge condition**.

Outline of the methods using a MOL approach

0. Start with Q^n_{MHD} and Q^n_{A} (the solution at t^n).
1. Build the right-hand sides of both semi-discrete systems, and independently update each system:

$$Q^*_{\mathrm{MHD}} = Q^n_{\mathrm{MHD}} + \Delta t\, \mathcal{L}(Q^n_{\mathrm{MHD}}),$$
$$Q^{n+1}_{\mathrm{A}} = Q^n_{\mathrm{A}} + \Delta t\, \mathcal{H}(Q^n_{\mathrm{A}}, \mathbf{u}^n),$$

   where $Q^*_{\mathrm{MHD}} = (\rho^{n+1}, \rho\mathbf{u}^{n+1}, \mathcal{E}^{n+1}, \mathbf{B}^*)$. $\mathbf{B}^*$ is the predicted magnetic field that in general does not satisfy a discrete divergence-free constraint.
2. Replace $\mathbf{B}^*$ by a 4^{th}-order-accurate discrete curl of the magnetic potential Q^{n+1}_{A}:

$$\mathbf{B}^{n+1} = \nabla \times Q^{n+1}_{\mathrm{A}}.$$

4^{th}-order 10-stage SSP-RK method is used as time integrator

4^{th}-order accurate in space and time for smooth Alfvén wave problem

Low-Storage SSP-RK4 Time-stepping

$Q^{(1)} = Q^{n}$;
$Q^{(2)} = Q^{n}$;
for $i = 1 : 5$ **do**
    $Q^{(1)} = Q^{(1)} + \frac{\Delta t}{6}\,\mathcal{L}\left(Q^{(1)}\right)$;
end
$Q^{(2)} = \frac{1}{25}\,Q^{(2)} + \frac{9}{25}\,Q^{(1)}$;
$Q^{(1)} = 15\,Q^{(2)} - 5\,Q^{(1)}$;
for $i = 6 : 9$ **do**
    $Q^{(1)} = Q^{(1)} + \frac{\Delta t}{6}\,\mathcal{L}\left(Q^{(1)}\right)$;
end
$Q^{n+1} = Q^{(2)} + \frac{3}{5}\,Q^{(1)} + \frac{\Delta t}{10}\,\mathcal{L}\left(Q^{(1)}\right)$;

This integrator coupled with FD-WENO is stable and 'TVD' up to CFL $= 3.07$

[Ketcheson, 2008]

The evolution equation of $\mathbf{A}$

$$\mathbf{A}_{,t} + (\nabla \times \mathbf{A}) \times \mathbf{u} = 0$$

can be rewritten as,

$$\mathbf{A}_{,t} + N_1\,\mathbf{A}_{,x} + N_2\,\mathbf{A}_{,y} + N_3\,\mathbf{A}_{,z} = 0,$$

where

$$N_1 = \begin{bmatrix} 0 & -u^2 & -u^3 \\ 0 & u^1 & 0 \\ 0 & 0 & u^1 \end{bmatrix}, N_2 = \begin{bmatrix} u^2 & 0 & 0 \\ -u^1 & 0 & -u^3 \\ 0 & 0 & u^2 \end{bmatrix}, N_3 = \begin{bmatrix} u^3 & 0 & 0 \\ 0 & u^3 & 0 \\ -u^1 & -u^2 & 0 \end{bmatrix}.$$

The flux Jacobian matrix in some arbitrary direction $\mathbf{n} = (n^1, n^2, n^2)$:

$$n^1 N_1 + n^2 N_2 + n^3 N_3 = \begin{bmatrix} n^2u^2 + n^3u^3 & -n^1u^2 & -n^1u^3 \\ -n^2u^1 & n^1u^1 + n^3u^3 & -n^2u^3 \\ -n^3u^1 & -n^3u^2 & n^1u^1 + n^2u^2 \end{bmatrix}.$$

The evolution equation of $\mathbf{A}$

The eigenvalues of the above matrix are

$$\lambda^1 = 0, \quad \lambda^2 = \lambda^3 = \mathbf{n} \cdot \mathbf{u},$$

The matrix of right eigenvectors can be written as

$$R = \left[\begin{array}{c|c|c} r^{(1)} & r^{(2)} & r^{(3)} \end{array} \right] = \begin{bmatrix} n^1 & n^2u^3 - n^3u^2 & u^1(\mathbf{u}\cdot\mathbf{n}) - n^1\|\mathbf{u}\|^2 \\ n^2 & n^3u^1 - n^1u^3 & u^2(\mathbf{u}\cdot\mathbf{n}) - n^2\|\mathbf{u}\|^2 \\ n^3 & n^1u^2 - n^2u^1 & u^3(\mathbf{u}\cdot\mathbf{n}) - n^3\|\mathbf{u}\|^2 \end{bmatrix}.$$

Assume that $\|\mathbf{u}\| \neq 0$ and $\|\mathbf{n}\| = 1$

$$\det(R) = -\|\mathbf{u}\|^3 \cos(\alpha)\sin(\alpha),$$

where α is the angle between $\mathbf{n}$ and $\mathbf{u}$.
There exist four degenerate directions, $\alpha = 0, \pi/2, \pi,$ and $3\pi/2$, in which the eigenvectors are incomplete. Hence, the new system is only *weakly hyperbolic*.

A special case – 2D

The divergence-free condition becomes

$$\nabla \cdot \mathbf{B} = B^1_{,x} + B^2_{,y} = 0$$

The magnetic field becomes

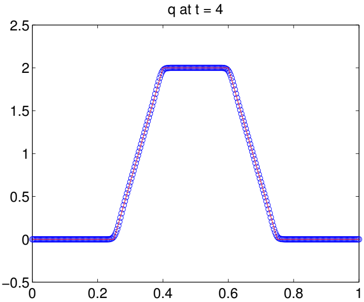
$$B^1 = A^3_{,y} \quad \text{and} \quad B^2 = -A^3_{,x}$$

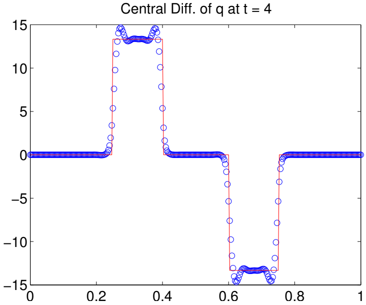
Only need to solve a *scalar potential* A^3

$$A^3_{,t} + u^1 A^3_{,x} + u^2 A^3_{,y} = 0$$

This case is *strongly hyperbolic.*

1D Advection Equation $q_{,t} + q_{,x} = 0$





oscillatory derivative by WENO-HCL

WENO-HCL scheme

Reconstruction applied to waves $q \Rightarrow$ non-oscillatory solution

[Christlieb, Rossmanith & Tang, 2013]

New Limiting Strategy

Our goal is:

To control the oscillation in the derivative $(\mathbf{B})$ when computed by

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Limiting Strategy:

Reconstruction applied to wave derivatives $q_{,x} \Rightarrow$
non-oscillatory derivative $(\mathbf{B})$

WENO for 1D Hamiltion–Jacobi

Consider a 1D Hamilton–Jacobi equation

$$q_{,t} + H\left(t, x, q, q_{,x}\right) = 0,$$

WENO-HJ scheme has the form:

$$\frac{dq_{ij}(t)}{dt} = -\hat{H}\left(t, x_i, q_i, q^-_{,xi}, q^+_{,xi}\right),$$

where $\hat{H}$ is the *numerical Hamiltonian* and

$$q^-_{,xi} = \Phi_{\text{WENO5}}\left(\frac{\Delta^+ q_{i-3}}{\Delta x}, \frac{\Delta^+ q_{i-2}}{\Delta x}, \frac{\Delta^+ q_{i-1}}{\Delta x}, \frac{\Delta^+ q_i}{\Delta x}, \frac{\Delta^+ q_{i+1}}{\Delta x}\right),$$

$$q^+_{,xi} = \Phi_{\text{WENO5}}\left(\frac{\Delta^+ q_{i+2}}{\Delta x}, \frac{\Delta^+ q_{i+1}}{\Delta x}, \frac{\Delta^+ q_i}{\Delta x}, \frac{\Delta^+ q_{i-1}}{\Delta x}, \frac{\Delta^+ q_{i-2}}{\Delta x}\right),$$

where

$$\Delta^+ q_{ij} := q_{i+1j} - q_{ij},$$

[Jiang & Peng, 2000]

Numerical Hamiltonian

$\hat{H}$ is typically *Lipschitz continuous* and consistent with H:

$$\hat{H}\left(t, x, q, u, u\right) = H\left(t, x, q, u\right),$$

E.g., a Lax-Friedrichs-type definition:

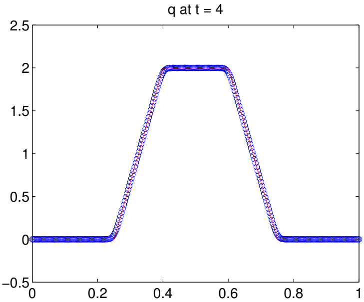
$$\hat{H}\left(t, x, q, q_x^-, q_x^+\right) = H\left(t, x, q, \frac{q_x^- + q_x^+}{2}\right) - \alpha\left(\frac{q_x^+ - q_x^-}{2}\right),$$

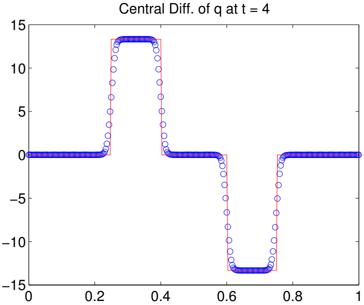
where

$$\alpha = \max_{u \in I(q_x^-, q_x^+)} \left|H_{,u}(t, x, q, u)\right|,$$

where $I(q_x^-, q_x^+)$ is the interval between q_x^- and q_x^+.

1D Advection Equation $q_{,t} + q_{,x} = 0$





non-oscillatory derivative by WENO-HJ

[Christlieb, Rossmanith & Tang, 2013]

A^3 in 2D case

In 2D

$$A^3_{,t} + u^1(x,y)A^3_{,x} + u^2(x,y)A^3_{,y} = 0$$

spatial discretization:

$$u^1_{ij}A_{,xij} = u^1_{ij}\left(\frac{A^{3+}_{,xij} + A^{3-}_{,xij}}{2}\right) - \alpha\left(\frac{A^{3+}_{,xij} - A^{3-}_{,xij}}{2}\right)$$

where

$$\alpha = \max_{i,j}\left|u^1_{ij}\right|$$

$$A^{3-}_{,xij} = \Phi_{\text{WENO5}}\left(\frac{\Delta^+ A^3_{i-3j}}{\Delta x}, \frac{\Delta^+ A^3_{i-2j}}{\Delta x}, \frac{\Delta^+ A^3_{i-1j}}{\Delta x}, \frac{\Delta^+ A^3_{i,j}}{\Delta x}, \frac{\Delta^+ A^3_{i+1,j}}{\Delta x}\right),$$
$$A^{3+}_{,xij} = \Phi_{\text{WENO5}}\left(\frac{\Delta^+ A^3_{i+2j}}{\Delta x}, \frac{\Delta^+ A^3_{i+1j}}{\Delta x}, \frac{\Delta^+ A^3_{ij}}{\Delta x}, \frac{\Delta^+ A^3_{i-1j}}{\Delta x}, \frac{\Delta^+ A^3_{i-2j}}{\Delta x}\right).$$

A in 3D case

$$
\begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix}_{,t} + \begin{bmatrix} 0 & -u^2 & -u^3 \\ 0 & u^1 & 0 \\ 0 & 0 & u^1 \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix}_{,x} + \begin{bmatrix} u^2 & 0 & 0 \\ -u^1 & 0 & -u^3 \\ 0 & 0 & u^2 \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix}_{,y}
+ \begin{bmatrix} u^3 & 0 & 0 \\ 0 & u^3 & 0 \\ -u^1 & -u^2 & 0 \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix}_{,z} = 0
$$

E.g.

$$
A^1_{,t} - u^2 A^2_{,x} - u^3 A^3_{,x} + u^2 A^1_{,y} + u^3 A^1_{,z} = 0
$$

Main idea:

WENO-HJ + Reconstruction applied to the derivatives $A^2_{,x}$ and $A^3_{,x}$

E.g.,

$$
u^2 A^2_{,x} = u^2 \left(\frac{A^{2+}_{,x} + A^{2-}_{,x}}{2} \right)
$$

Artificial resistivity

E.g.,

$$A^1_{,t} - u^2 A^2_{,x} - u^3 A^3_{,x} + u^2 A^1_{,y} + u^3 A^1_{,z} = \varepsilon^1 A^1_{,x,x}$$

ε^1 is evaluated by

$$\varepsilon^1 = 2\nu\gamma^1 \frac{\Delta x^2}{\Delta t},$$

where γ^1 is the smoothness indicator of A^1, and ν is a footnotesize constant. The smoothness indicator γ^1 is computed by:

$$\gamma^1_{ijk} = \left| \frac{a^-}{a^- + a^+} - \frac{1}{2} \right|,$$

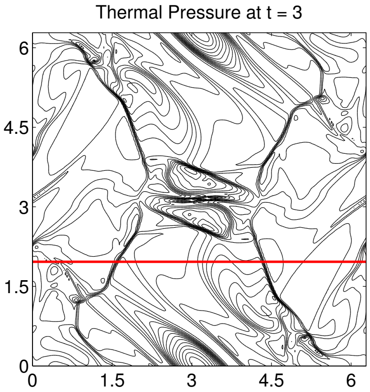
where

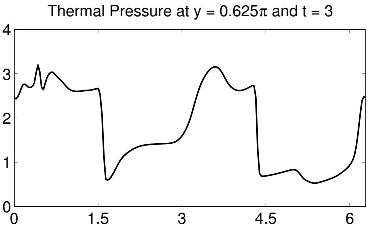
$$a^- = \left\{\epsilon + \left(\Delta x\, A^{1-}_{,xijk}\right)^2\right\}^{-2} \quad a^+ = \left\{\epsilon + \left(\Delta x\, A^{1+}_{,xijk}\right)^2\right\}^{-2},$$

Features of our schemes

- There is no mesh staggering.
- It is an explicit MOL approach.
- No spatial integration or multidimensional reconstructions
- $\mathbf{B}$ satisfies a discrete divergence-free condition exactly.
- All the quantities $(\rho, \rho\mathbf{u}, \mathcal{E}, \mathbf{B})$ are conserved.
- Oscillations in the solutions are controlled (including $\mathbf{B}$).

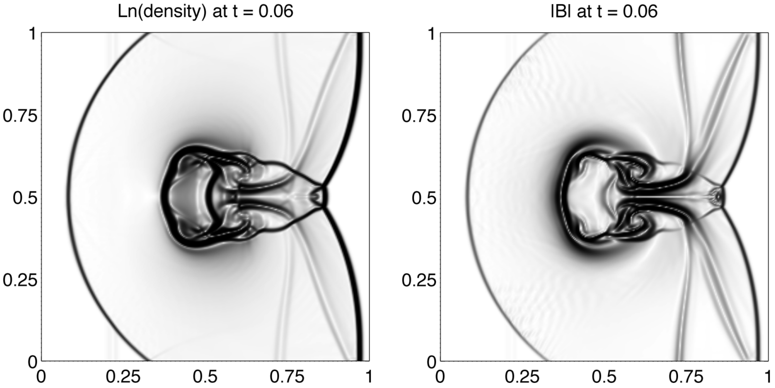
Orszag-Tang Vortex





The simulation can run to $t = 30$ and no significant oscillations.

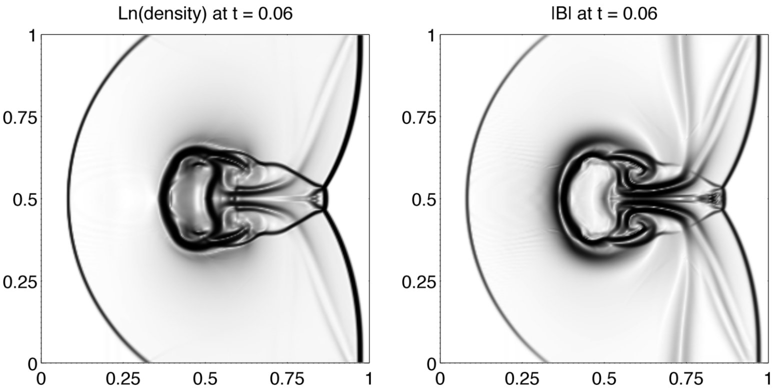
2D Cloud-Shock Interaction



The resolution is 256×256.

[Christlieb, Rossmanith & Tang, 2013]

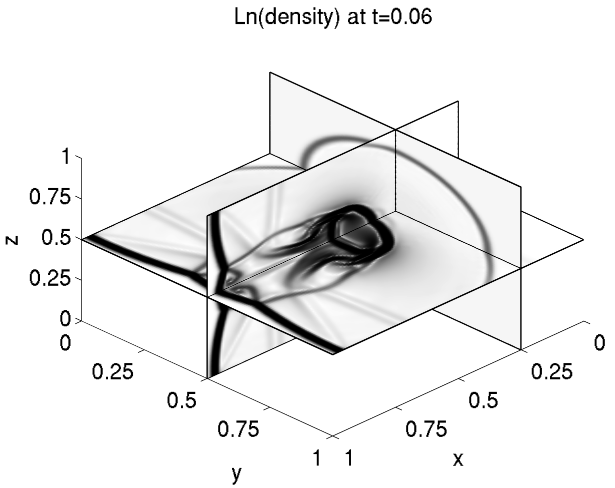
2D Cloud-Shock Interaction



The resolution is 256×256.
[Rossmanith, 2006]

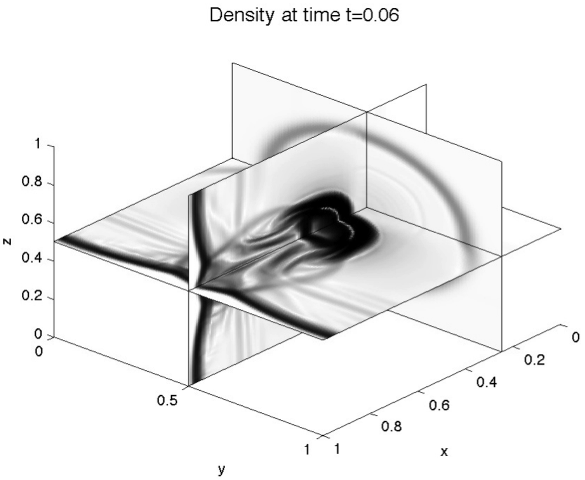
[Rossmanith, MHDCLAW, Available from http://www.public.iastate.edu/~rossmani/claw/MHDCLAW]

3D Cloud-Shock Interaction



The resolution is 128^3 [Christlieb, Rossmanith & Tang, 2013]

3D Cloud-Shock Interaction



The resolution is 150^3 [Helzel, Rossmanith, & Taetz, 2013]

Conclusion and Future Work

- The WENO CT methods have been successfully used to solve 2D/3D ideal MHD problems.
- It is high-order and efficient. (E.g. vs 3$^{\text{rd}}$-order unsplit FV code)
- There is evidence that a high-order scheme can obtain some structure with fewer grid points.
- Future works include the extension to the AMR and positivity preserving.

Thank you!

Reference

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