Data Driven Observations of System Equilibration

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Abstract

In near-vacuum-hohlraum inertial confinement fusion experiments, the ion's mean free path is too long for the system to be sufficiently described by the hydrodynamic equations [1]. To simulate such systems, extended moment hydrodynamic models are employed [2]. Interestingly, the additional moments include dissipative processes that return the system to equilibrium where fewer moments are sufficient. An obvious reduced order model is one that reduces the extended moments as the system equilibrates. To do this well, the reduced order model must identify when fewer moments will suffice. In this study, we use data driven techniques to observe Grad's 13 moment hydrodynamic equations reduce to Navier-Stokes (N-S) 5 moment description [3].

Our computational tools come from the emerging field of data driven dynamical systems. Modern algorithms, rooted in the Koopman operator framework, approximate nonlinear systems of equations as linear ones. In this study, we employ dynamic mode decomposition (DMD) to observe our dynamical system equilibrate to its slow manifold [4]. Moreover, we supplement our DMD findings by using dimension reduction techniques to observe equilibration in an ensemble of simulations; this offers a clear visualization of the equilibration process.

Moment reduction in Grad's eqns

At linear order, expanding Grad's hydrodynamics about equilibrium (*i.e.* $\varepsilon \to 0$) produces the N-S equations [3]. This expansion process can be interpreted as computing Chapman-Enskog closures (*e.g.* $\sigma = -\frac{4}{3}\nabla_x u$ and $q = -\frac{5}{2}\partial_x T$). In this work we consider the Fourier transform $\rho(x,t) = \sum_{-\infty}^{+\infty} \rho_{\boldsymbol{k}} e^{ikx}$ of the these PDE's.

Grad's Eqns

$$\partial_t \begin{bmatrix} \rho \\ u \\ p \\ \sigma \\ q \end{bmatrix} = - \begin{bmatrix} \partial_x u \\ \partial_x (p + u + \sigma) \\ (2/3)\, \partial_x (u + q) \\ \partial_x (2u + 4/5 q) - \sigma/\epsilon \\ \partial_x (5T/2 + \sigma) - 2q/3\epsilon \end{bmatrix}$$

Fourier →

$$\begin{aligned} \partial_t \rho_k &= -ik\, u_k \\ \partial_t u_k &= -i\, k\, (\rho_k + T_k + \sigma_k) \\ \partial_t T_k &= -ik \tfrac{2}{3} (u_k + q_k) \\ \partial_t \sigma_k &= -\tfrac{4}{3} ik u_k - \tfrac{8}{15} ik q_k - \tfrac{1}{\epsilon} \sigma_k \\ \partial_t q_k &= -\tfrac{5}{2} ik T_k - ik\sigma_k - \tfrac{2}{3\epsilon} q_k \end{aligned}$$

$q \approx -\epsilon \frac{15}{4} \nabla_x T \quad \downarrow \quad \sigma \approx -\epsilon \frac{4}{3} \nabla_x u$

N-S Eqns

$$\partial_t \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = - \begin{bmatrix} \partial_x u \\ \partial_x (p + u) - \epsilon \frac{4}{3} \partial_x^2 u \\ \frac{2}{3} \partial_x u - \epsilon \frac{5}{2} \partial_x^2 T \end{bmatrix}$$

Fourier →

$$\begin{aligned} \partial_t \rho_k &= -ik\, u_k \\ \partial_t u_k &= -i\, k\, (\rho_k + T_k + \sigma_k) \\ \partial_t T_k &= -ik \tfrac{2}{3} (u_k + q_k) \end{aligned}$$

Observation of dimension reduction in Grad's eqns

In Figure 1, we demonstrate that after equilibration N-S equations are a reduced model of Grad's moment equations. In Figure 2, we plot Rowes and Saul's inverse reconstruction error [5] to observe principal component analysis' (PCA) ability to reduce the ensemble's dimension. We demonstrate that the data can reduce from 10 (5 complex values) to 6 dimensions at late times; which matches the reduction from Grad's moments ($\rho_k, u_k, T_k, \sigma_k, q_k$ each is a complex value) to N-S moments (ρ_k, u_k, T_k).

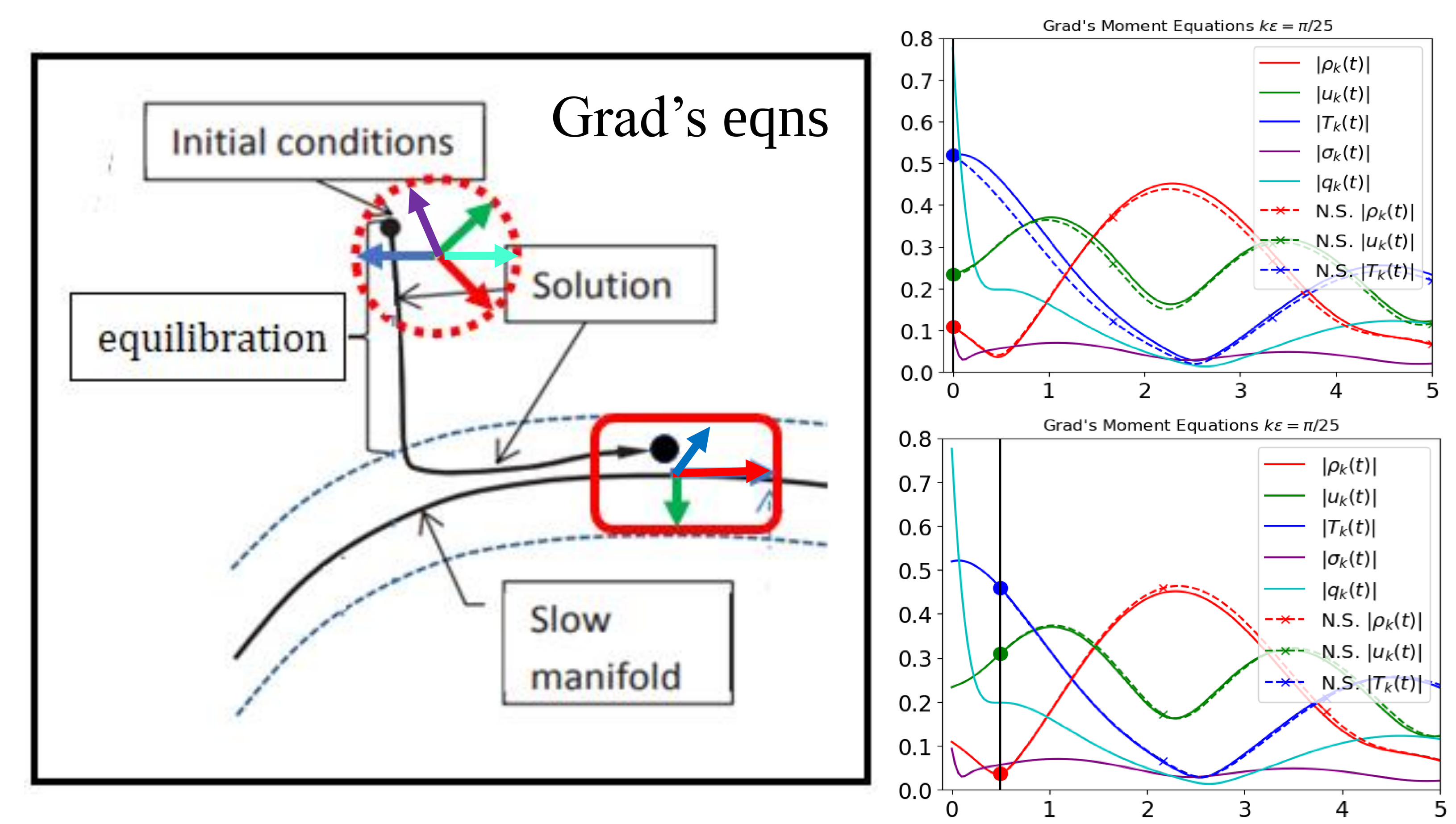

Figure 1 – Left: Representation of an ensemble ($N = 10{,}000$) of simulations equilibrating according to Grad's eqns towards a slow manifold (image altered from [3]). **Right:** A sample trajectory from the ensemble. We initialize N-S eqns to the Grad's moment values at time t = 0 (top) and $t = 0.5$ (bottom); the black bar marks initial conditions..

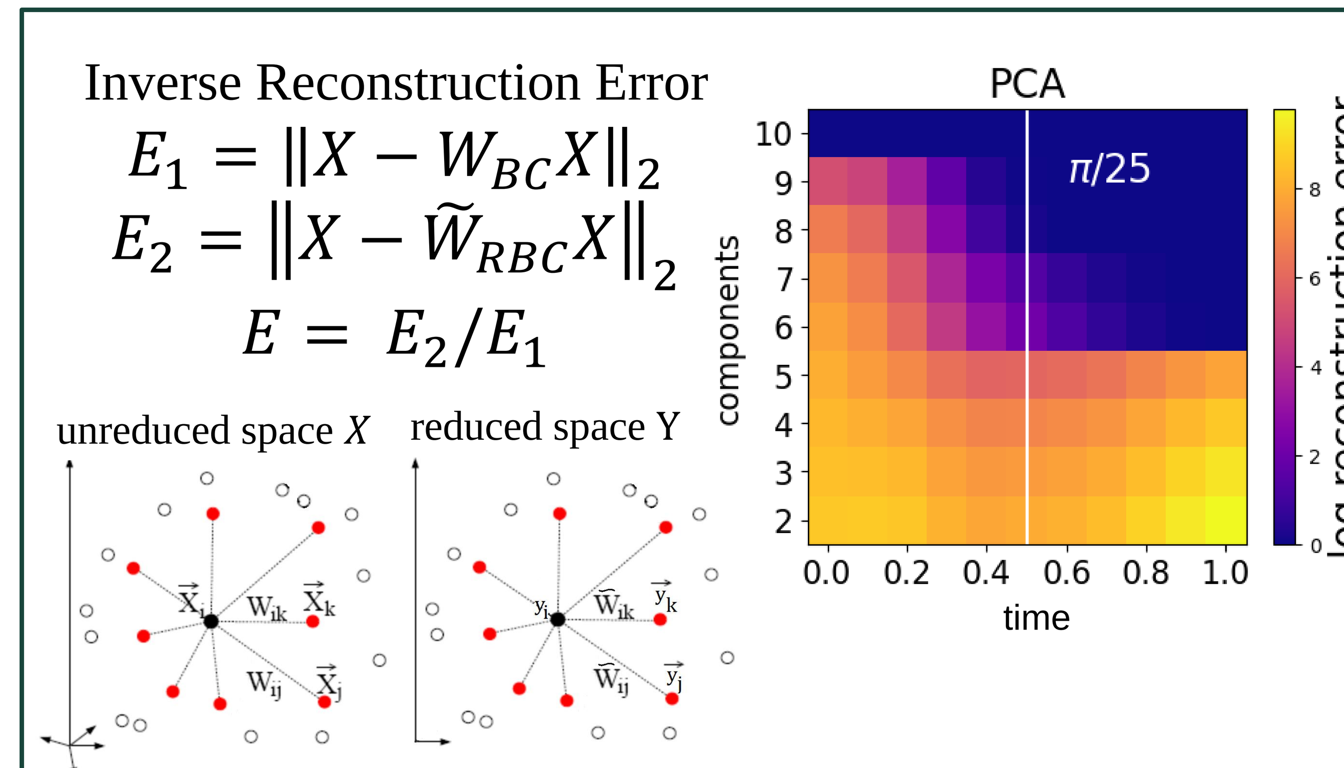

Figure 2 – Left (top): Formulation of the reconstruction error. W_{BC} and $\widetilde{W}_{RBC}$ are barycenter weight matrices in either the full coordinates X or PCA reduced coordinates Y. **Left (bottom):** Visualization of the row elements of each barycenter weight matrix. **Right:** Plot of the reconstruction error. We vary the quantity of PCA basis vectors along the y-axis and the time along the x-axis.

Identification of the slow manifold in Grad's eqns

In Hartman-Grobman theory, the Jacobian of a non-linear system is used near an equilibrium to linearize. As an improvement, the Koopman operator can be used away from equilibrium to linearize the dynamics. Like the Jacobian, the eigensystem of the Koopman operator characterizes the slow manifold [6]. DMD is a known way to discover a finite dimensional representation of the Koopman operator, K and its eigenvalues Λ and vectors Ψ. We conduct DMD on a window of the data (*i.e.* subsection of the sequential data represented as a blue band in Figure 3: left). Windowed DMD produces a timeseries of eigenvalues, see in Figure 3: right. We observe that two of the eigenvalues have an absolute value that differs by many orders of magnitude by $t = 0.5$, which is a similar time scale as seen in the dimension reduction plots. Furthermore, plots of the similarity between the DMD eigenvectors at time t_i and t_j are presented in Figure 4. We observe that after $t = 0.5$, DMD rediscovers the same eigenvector; this indicates that the slow manifold has been identified.

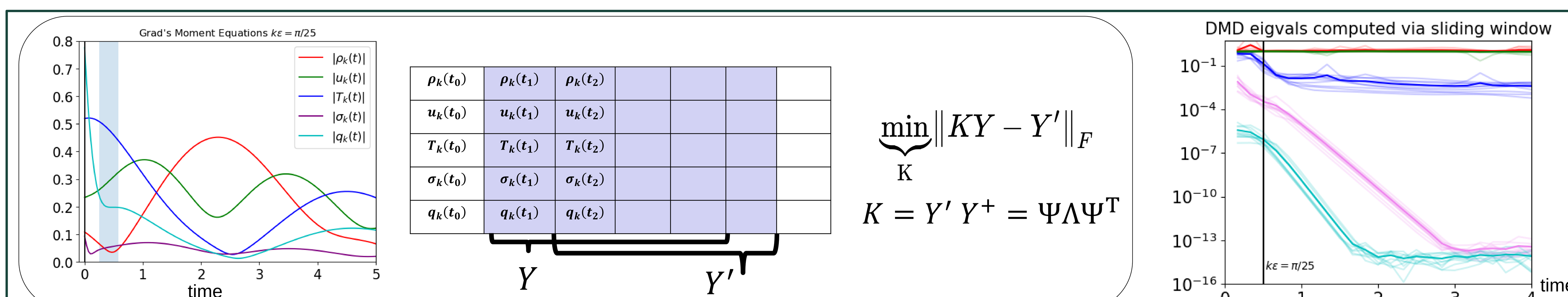

Figure 3 – Left: Description of DMD on a sliding window. Blue band represents the sliding window. **Right:** timeseries of DMD eigenvalues.

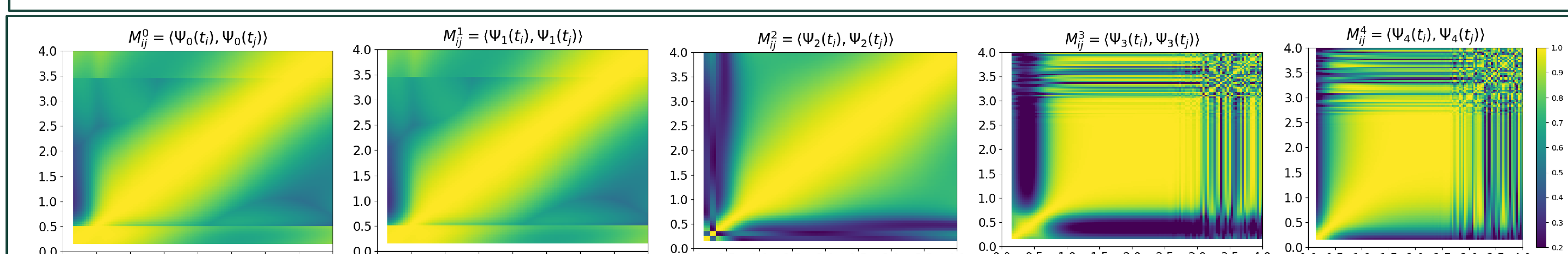

Figure 4: Plot of the similarity between DMD eigenvectors across the timeseries. Similarity measured by complex dot product.

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[1] Rinderknecht, Hans G., et al. *Plasma Physics and Controlled Fusion* 60.6 (2018): 064001.
[2] Hoffman, Nelson M., et al. *Physics of Plasmas* 22.5 (2015).
[3] Gorban, Alexander, and Ilya Karlin. *Bulletin of the American Mathematical Society* 51.2 (2014): 187-246.
[4] Schmid, Peter J. *Journal of fluid mechanics* 656 (2010): 5-28.
[5] Rowes, Sam T., and Lawrence K. Saul. *science* 290.5500 (2000): 2323-2326.
[6] Brunton, Steven L., et al. arXiv preprint arXiv:2102.12086 (2021).